Undergrad PDE HW9 Junyan Zhang

[5.8] 1

$$\begin{cases} \cancel{\frac{\partial u}{\partial t}} \partial_t u = k\partial_x^2 u \\ u(0,t)=0,\ \partial_x u(L,t) = -hu(L,t) \\ u(x,0)=f(x) \end{cases}$$

(2) Solve if $hL=-1$

Sol: Let $u(x,t) = X(x)T(t)$.

then we get $\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0 \\ X'(L) = -hX(L) \end{cases}$ $T'(t)+\lambda k T(t) = 0$

$\Downarrow$

$T_n(t) = T_n(0)e^{-\lambda_n kt}$

$\Rightarrow X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$\because X(0)=0 \Rightarrow C_1=0$

$\Rightarrow X(x) = C_2\sin(\sqrt{\lambda}x)$

$X'(L) = -hX(L)$

$\Rightarrow \sqrt{\lambda}\cos(\sqrt{\lambda}L) = -h\,\overset{\sin}{\cancel{\cos}}(\sqrt{\lambda}L)$

$\Rightarrow \tan(\sqrt{\lambda}L) = -\frac{\sqrt{\lambda}}{h} = \sqrt{\lambda}L$

$\Rightarrow \lambda_n \sim \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ (n large)

$X_n(x) = \sin\left(\frac{(n+\frac{1}{2})\pi}{L}x\right)$

~~So u(x,t) =~~

$u(x,t) \approx \sum_{n\geq 1} e^{-\lambda_n kt}\, T_n(0)\, X_n(x)$

Then plug in $t=0$ we can solve $T_n(0)$ from $f(x)$.

$\square$

8. Consider $\begin{cases} \frac{d^2\phi}{dx^2}+\lambda\phi = 0 \\ \phi(0) = \frac{d\phi}{dx}(0) \\ \phi(1) = -\frac{d\phi}{dx}(1) \end{cases}$

(a) Show that $\lambda > 0$.

Proof: By Rayleigh quotient,

$$\lambda = \frac{-\phi\phi'\Big|_0^1 + \int_0^1 |\phi'(x)|^2\,dx}{\int |\phi|^2}\,dx \geq 0.$$

If $\lambda = 0$ then $\phi(0) = \phi(1) = 0 \Rightarrow \phi'(0) = 0 \Rightarrow \phi$ is identically 0

so $\lambda > 0$

$\square$

(2). Prove that eigenfunction corresponding to different eigenvalues are orthogonal.

<u>Pf</u>: $L(\phi) = \phi''(x)$

$$L\phi = \lambda\phi$$

$$\text{so } \int_0^1 (\phi_n L(\phi_m) - L(\phi_n)\phi_m)\,dx \overset{\rceil}{=} (\lambda_n - \lambda_m)\int_0^1 \phi_n \phi_m$$

$$= (\phi_n \phi_m' - \phi_m \phi_n')\Big|_0^1 = 0$$

so for distinct eigenvalues λ_n, λ_m, $\int_0^1 \phi_n \phi_m = 0$.
$\Rightarrow$ orthogonal $\square$

(3) Show that. $\tan\sqrt{\lambda} = \dfrac{2\sqrt{\lambda}}{\lambda - 1}$.

$\lambda > 0 \Rightarrow \phi(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$.

Combining with boundary conditions, we have

$$C_1 = C_2\sqrt{\lambda},$$

$$C_1\cos\sqrt{\lambda} + C_2\sin\sqrt{\lambda} = C_1\sqrt{\lambda}\sin\sqrt{\lambda} - C_2\sqrt{\lambda}\cos\sqrt{\lambda}.$$

eliminate C_1

$$\Rightarrow C_2\left((\lambda - 1)\sin\sqrt{\lambda} - 2\sqrt{\lambda}\cos\sqrt{\lambda}\right) = 0$$

$$\Rightarrow \tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1} \to 0 \text{ as } \lambda \to \infty.$$

(4) Solve
$$\begin{cases} \partial_t u = \partial_x^2 u \\ u(0,t) = u_x(0,t) \\ u(1,t) + \partial_x u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

<u>Sol</u>: Let $u(x,t) = X(x)T(t)$

$\Rightarrow T_n(t) = e^{-\lambda_n t}$

eigenfunctions

$$u_n(x,t) = \sum_{n \geq 1} A_n X_n(x) e^{-\lambda_n t}$$

$$u(x,0) = f(x) \Rightarrow A_n = \frac{\int_0^1 f X_n\,dx}{\int_0^1 X_n^2(x)\,dx}$$

9. Consider $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$
$$\begin{cases} \phi(0) = \frac{d\phi}{dx}(0) & \phi(1) = \beta\frac{d\phi}{dx}(1) \end{cases}$$
For what values of β is $\lambda = 0$ an eigenvalue

In fact one can solve ϕ (a linear functions) when $\lambda = 0$ and get $\beta = 2$.

Sol: From Rayleigh quotient,
$$\lambda = \frac{-\phi'(1)\phi(1) + \phi'(0)\phi(0) + \int_0^1 (\phi')^2 dx}{\int_0^1 \phi^2 dx}.$$

$$\lambda = 0 \text{ iff } -\phi'(1)\phi(1) + \phi'(0)\phi(0) + \int_0^1 (\phi')^2 dx = 0$$

$$\phi(0) = \phi'(0) \quad \phi(1) = \beta\phi'(1)$$

$$\Rightarrow \beta\phi'(1)^2 = \phi'(0)^2 + \int_0^1 (\phi')^2 dx$$

$$\Rightarrow \beta = \frac{(\phi'(0))^2 + \int_0^1 (\phi')^2 dx}{(\phi'(1))^2}$$

$\square$

[5.9] 1. $\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + (\lambda\sigma(x) + q(x))\phi = 0$.

Estimate the large eigenvalues

(1) $\phi'(0) = 0$. $\phi'(L) = 0$

$$\phi(x) \sim (\sigma p)^{-\frac{1}{4}} \cos\left(\sqrt{\lambda} \int_0^x \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx_0\right)$$

$$\frac{d\phi}{dx}(L) = 0 \Rightarrow -(\sigma p)^{-\frac{1}{4}} \sin\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx_0\right) \lambda^{\frac{1}{2}} \left(\frac{\sigma}{p}\right)^{\frac{1}{2}}\Big|_{x=L} = 0$$

$$\Rightarrow \sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx_0 \approx n\pi$$

$$\Rightarrow \lambda \sim \cancel{\int_0^L \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx_0} \left(\frac{n\pi}{\int_0^L \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx}\right)^2$$

(3) Use the same method as in (1)

$$\lambda_n \sim \left(\frac{(n+\frac{1}{2})\pi}{\int_0^L \left(\frac{\sigma}{p}\right)^{\frac{1}{2}} dx_0} \right)^2$$

$\square$

[5.10] 2. Obtain a formula for an infinite series using Parseval

(b) Fourier sine Series of $f(x)=1$ on $[0, L]$

The Fourier series is

$$1 \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \qquad a_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

By Parseval's identity, we have

$$L = \sum_{n \text{ odd}} \left(\frac{4}{n\pi}\right)^2 \cdot \frac{L}{2} = \sum_{n=1}^{\infty} \frac{8L}{(2n-1)^2 \pi^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

3. Consider $f(x)$ on $[a, b]$. Approximate $f(x)$ by a const.

Show that the best const is the average of $f(x)$ in $[a, b]$

Pf: Let $E(t) = \int_a^b |f(x) - t|^2 dx$

$$E'(t) = 0 \Rightarrow t_0 = \frac{1}{b-a} \int_a^b f(x)\, dx$$

and $E''(t_0) > 0$. $\square$

Rmk: The background is: Best approximator in L^2-norm is Fourier series (Bessel ineq)

If we restrict the approximator to be a const, then it means we only ~~use the c.~~ approximate ~~from~~ $f(x)$ by the first function in the orthonormal basis $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$ which is the const term in Fourier series. Only this $\rightarrow$ 1