Heat.

Sound
Elastic strings, membranes
Waves

Chapter 4: Wave Equations

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Solution to the IBVP?

$$\partial_{tt} u = \kappa \partial_{xx} u + Q(x,t), \quad \text{with } x \in (0,L), t \geq 0$$

IC: $u(x,0) = f(x), \partial_t u(x,0) = g(x)$

BC: $u(0,t) = \phi(t), u(L,t) = \psi(t)$

Section 4.2: Derivation of a vertically vibrating string
Section 4.3: Boundary conditions
Section 4.4: Vibrating string with fixed ends
Section 4.5: Vibrating membrane

Outline

Section 4.2: Derivation of a vertically vibrating string

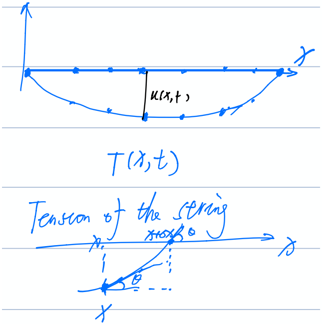
Section 4.3: Boundary conditions

Section 4.4: Vibrating string with fixed ends

Section 4.5: Vibrating membrane

Section 4.2: Derivation of a vertically vibrating string

Consider a vibrating string:



The string is on $x \in [0, L]$:

- Mass density $\rho(x)$, body force $Q(x,t)$
- Motion is entirely vertical (perfectly elastic string)
- Vertical displacement $u(x,t)$: small

Section 4.2: Derivation of a vertically vibrating string

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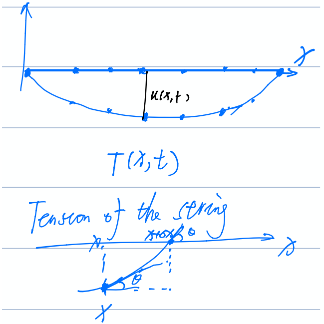

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Newton's Law

$$F = ma = \rho(x)\Delta x \partial_{tt} u$$

$$F = \text{force+ tension } = \rho(x)\Delta x Q(x,t) +$$

$$+ T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t)$$

$$\Rightarrow \rho(x)\partial_{tt} u = \partial_x[T(x,t)\sin\theta(x,t)] + \rho(x)Q$$

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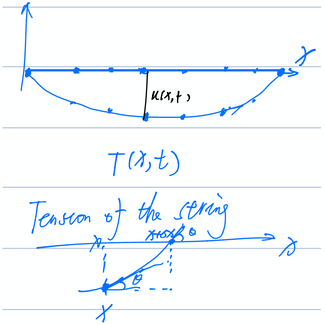

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Note: tangent slope $= \partial_x u = \tan\theta(x,t) = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$ when θ small $\Downarrow$

$$\rho(x)\partial_{tt} u = \partial_x[T(x,t)\partial_x u] + \rho(x)Q$$

If $\rho(x) \equiv \rho_0, T(x,t) \equiv T_0$ and $Q(x,t) = 0$, then

$$\partial_{tt} u = c^2 \partial_{xx} u, \quad c^2 = \frac{T_0}{\rho_0} \quad \text{WAVE equation}$$

$$\rho(x)\partial_{tt}u = T_0\partial_{xx}u + \rho(x)Q(x,t) \qquad (4.2.7)$$

$$\rho(x)\partial_{tt}u = T_0\partial_{xx}u \qquad (4.2.9)$$

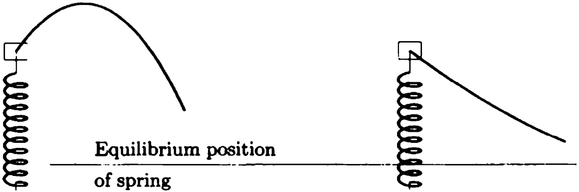
4.2.1. (a) Using Equation (4.2.7), compute the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. The boundary conditions are $u(O) = 0$ and $u(L) = 0$.

(b) Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies (4.2.9).

Outline

Section 4.3: Boundary conditions

Question: which massless string is in space (or on earth)?



Honey in space (Youtube)

More interesting: the spring's base vibrates

- $u(0,t) = y(t)$, $u(L,t) = 0$
- the spring: position $y(t) - y_s(t)$, equilibrium length l
- Newton + Hooke's laws $m\frac{d^2y}{dt^2} = -k(y - y_s(t) - l) +$ force

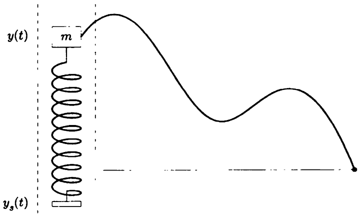


Figure 4.3.1 Spring-mass system with a variable support attached to a stretched string.

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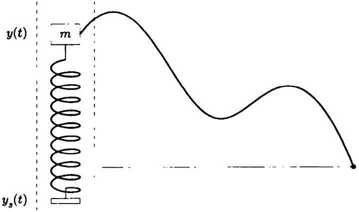

Figure 4.3.1 Spring-mass system with a variable support attached to a stretched string.

tension: $T(0,t)\sin\theta(0,t) \approx T(0,t)\tan\theta(0,t) = T(0,t)\partial_x u(0,t) \Rightarrow$

$$m\frac{d^2u(0,t)}{dt^2} = -k[u(0,t) - y_s(t) - l] + T(0,t)\partial_x u(0,t) + G(t)$$

- when the external force $G = mg$ with $g = 0$ and small mass $m = 0$:

$$k[u(0,t) - y_s(t) - l] = T(0,t)\partial_x u(0,t)$$

$\Rightarrow$ upwards if $[u(0,t) - y_s(t) - l] > 0$.

More interesting: the spring's base vibrates

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- the spring: position $y(t) - y_s(t)$, equilibrium length l
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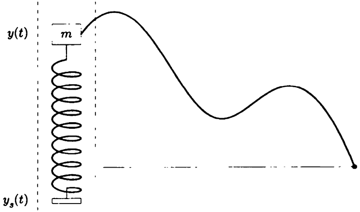

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$$m\frac{d^2u(0,t)}{dt^2} = -k[u(0,t) - y_s(t) - l] + T(0,t)\partial_x u(0,t) + G(t)$$

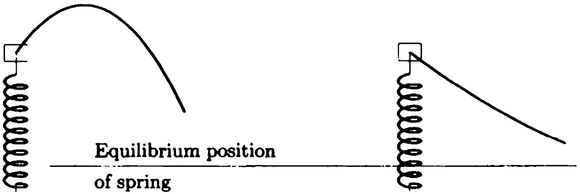
- when the external force $G = mg$ with $g = 0$ and small mass $m = 0$:

$$k[u(0,t) - y_s(t) - l] = T(0,t)\partial_x u(0,t)$$

$\Rightarrow$ upwards if $[u(0,t) - y_s(t) - l] > 0$.

Question: what about right end point $u(L,t)$?

Question: which massless string is in space (or on earth)?



Outline

Section 4.4: Vibrating string with fixed ends

How to solve the wave equation IBVP?

$$\partial_{tt} u = c^2 \partial_{xx} u,$$
$$u(0,t) = 0, u(L,t) = 0$$
$$u(x,0) = f(x), \partial_t u(x,0) = g(x)$$

Separation of variables

Section 4.4: Vibrating string with fixed ends

How to solve the wave equation IBVP?

$$\begin{aligned} \partial_{tt}u &= c^2\partial_{xx}u, \\ u(0,t) &= 0, u(L,t) = 0 \\ u(x,0) &= f(x), \partial_t u(x,0) = g(x) \end{aligned}$$

Separation of variables v.s. method of eigenfunction expansion
$u(x,t) = \sum_{n=0}^{\infty}[a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}] \sin\frac{n\pi x}{L}$

Is it a solution? conditions for TBTD

Uniqueness of solution?

Section 4.4: Vibrating string with fixed ends

$$\partial_{tt}u = c^2\partial_{xx}u,$$
$$u(0,t) = 0, u(L,t) = 0$$
$$u(x,0) = f(x), \partial_t u(x,0) = g(x)$$

Solution: $u(x,t) = \sum_{n=0}^{\infty}[a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}]\sin\frac{n\pi x}{L}$

Interpretation: musical stringed instruments

- normal modes of vibration: $[a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}]\sin\frac{n\pi x}{L}$
- sound intensity (amplitude): $\sqrt{a_n^2 + b_n^2}$
- circular frequency $n\pi c/L$ (# of oscillations in 2π unit time)

$$u(x,t) = \sum_{n=0}^{\infty}[a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}] \sin\frac{n\pi x}{L}$$

Standing wave: $\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$

Traveling wave: the standing wave = two traveling waves

$$\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L} = \frac{1}{2}\underbrace{\cos\frac{n\pi}{L}(x-ct)}_{\text{wave traveling to the right}} - \frac{1}{2}\underbrace{\cos\frac{n\pi}{L}(x+ct)}_{\text{wave traveling to the left}}$$

- Recall that: $cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos\frac{n\pi}{L}(x-ct)$: wave traveling to the right, with velocity c
  $\cos\frac{n\pi}{L}(x+ct)$: wave traveling to the left, with velocity c

Standing and traveling wave (Wiki)
Wave equation (Hojun Lee's simulation)

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

4.4.4. Redo Exercise 4.4.3(b) by the eigenfunction expansion method.

Question: Will there be a solution to the IBVP?

$$\partial_{tt}u = c^2\partial_{xx}u - \beta\partial_x u$$
$$u(0,t) = 0, u(L,t) = 0$$
$$u(x,0) = f(x), \partial_t u(x,0) = g(x)$$

Outline

Section 4.5: Vibrating membrane

$$\partial_{tt}u = c^2\nabla^2 u$$

2D wave equation: $\nabla^2 u = (\partial_{xx} + \partial_{yy})u$

- The derivation of the equation: similar to the 1D string.
- Boundary treatment (Stokes' theorem)

$$\int\int \nabla \times \mathbf{B} \cdot \mathbf{n} dA = \int \mathbf{B} \cdot \mathbf{t} ds$$

where dA is differential surface area, ds is diff. arc length.

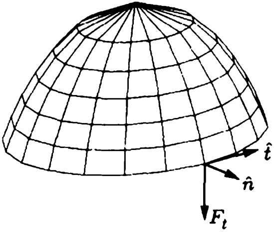


Figure 4.5.1 Perturbed stretched membrane with approximately constant tension T_0. The normal vector to the surface is $\hat{n}$ and the tangent vector to the edge is $\hat{t}$.