[2.] Define $f(x) = \pi - x$ for $0 < x < 2\pi$. $f(0) = f(2\pi) = 0$ and extend f to a 2π-periodic function on $\mathbb{R}$ (in the obvious way). Show that the Fourier series for f is $2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

Pf: $a_k = \frac{1}{\pi}\int_0^{2\pi} (\pi - t)\cos kt\,dt = 0$, $k \geq 0$

$b_k = \frac{1}{\pi}\int_0^{2\pi} (\pi - t)\sin kt\,dt$.

$= \frac{1}{k\pi}\int_0^{2\pi} (t-\pi)\,d\cos kt = \frac{1}{\pi k}\cdot\left[\cos kt\cdot(t-\pi)\Big|_0^{2\pi} - \int_0^{2\pi}\cos kt\,dt\right]$

$= \frac{1}{\pi k}\cdot[\pi - (-\pi)] = \frac{2}{k}$

Thus, the Fourier series $= 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$.

[3] Let $f \in BV[-\pi, \pi]$ with $f(-\pi) = f(\pi)$. Show that both $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ exists, and that each is at most $\frac{1}{n}V_{-\pi}^{\pi} f$

Pf: f is of bdded variation $\Rightarrow f$ is a Riem integrable function.

$f(x)\sin nx$ bdd variation $\Rightarrow f\sin nx$ is a Riem integrable function

NTS: $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \leq \frac{V_{-\pi}^{\pi} f}{n}$.

Let $-\pi = t_0 < \cdots < t_m = \pi$ be any partition of $[-\pi, \pi]$.

By MVT, there exists s_k s.t

$$\frac{\left(\left(-\frac{1}{n}\cos(nt_k)\right) - \left(-\frac{1}{n}\cos(nt_{k-1})\right)\right)}{t_k - t_{k-1}} = \sin(n s_k)$$

Consider the following $\overset{\text{Riem}}{\text{sum}}$. then we have.

$$\sum_{k=1}^{m} f(s_k)\sin(n\cdot s_k)\cdot(t_k-t_{k-1}) = \sum_{k=1}^{m} f(s_k)[-\tfrac{1}{n}\cos(nt_k)+\tfrac{1}{n}\cos(nt_{k-1})]$$

$$= f(s_1)\cdot\tfrac{1}{n}\cos(nt_0) - f(s_m)\cdot\tfrac{1}{n}\cos(nt_m) + \sum_{k=1}^{m-1}(f(s_k)-f(s_{k+1}))\cdot[-\tfrac{1}{n}\cos(nt_{k-1})]$$

$$\leq \tfrac{1}{n}|f(s_1)-f(s_m)| + \sum_{k=1}^{m-1}|f(s_k)-f(s_{k+1})|]$$

$$\leq \tfrac{1}{n}[|f(s_1)-f(-\pi)| + \sum_{k=1}^{m-1}|f(s_k)-f(s_{k+1})| + |f(s_m)-f(\pi)|$$

$$\leq \tfrac{1}{n} V_{-\pi}^{\pi} f$$

[6] Let $f:\mathbb{R}\to\mathbb{R}$ be 2π-periodic and Riem integrable on $[-\pi,\pi]$. Prove that $\lim_{x\to 0}\int_{-\pi}^{\pi}|f(x+t)-f(t)|^2dt=0$

Pf: Using Exercise 5. $\forall \varepsilon.\ \exists g\in C^{2\pi}$ s.t $\int_{-\pi}^{\pi}|f-g|^2<\varepsilon$.

Since g is uniformly cts over $[-\pi-1,\pi+1]$.

$\forall \varepsilon.\ \exists\delta$ s.t if $|x|<\min\{\delta,1\}$. $|g(x+t)-g(t)|<\sqrt{\frac{\varepsilon}{2\pi}}$ for $t\in[-\pi-1,\pi+1]$

$$\Rightarrow \int_{-\pi}^{\pi}|g(x+t)-g(t)|^2dt < \frac{\varepsilon}{2\pi}\cdot 2\pi<\varepsilon \text{ if } |x|<\min\{\delta,1\}$$

Then $\int_{-\pi}^{\pi}|f(x+t)-f(t)|^2 \leq 3\left(\int_{-\pi}^{\pi}|f(x+t)-g(x+t)|^2+\int_{-\pi}^{\pi}|g(x+t)-g(t)|^2\right.$

$$\left.+\int_{-\pi}^{\pi}|g(t)-f(t)|^2dt\right)$$

$$\leq 3\cdot 3\varepsilon = 9\varepsilon.$$

Since $\int_{-\pi}^{\pi}|f(x+t)-g(x+t)|^2=\int_{-\pi}^{\pi}|f(t)-g(t)|^2dt$

Hence, $\lim_{x\to 0}\int_{-\pi}^{\pi}|f(x+t)-f(t)|^2dt=0$.

7. Define $f(x) = (\pi - x)^2$ for $0 \le x \le 2\pi$ and extend f to a 2π-periodic continuous function on $\mathbb{R}$. Show the Fourier series for f is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$. Since the series is uniformly convergent, it actually converges to f. In particular, note that setting $x=0$ yields the familiar formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Pf: $a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx = \frac{4}{n^2}$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 dx = \frac{2\pi^2}{3}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx = 0$

Fourier series $= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Since the series is uniformly convergent $\Rightarrow$ $f = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Let $x=0$ $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

8. Fix $n \ge 1$. and $\varepsilon > 0$.

(a) show that there exists a cts function $f \in C^{2\pi}$ s.t $\|f\|_\infty = 1$ and

$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t) - \operatorname{sgn} D_n(t)| dt < \frac{\varepsilon}{n+1}$

(b) show $S_n(f)(0) \ge \lambda_n - \varepsilon$. hence $\|S_n(f)\|_\infty \ge \lambda_n - \varepsilon$.

(a). By Thm 14.9 or Exercise 5. (since. $\operatorname{sgn} D_n(t)$ is 2π-Riem-integrable function)

we can find such an $f(t)$.

(b) $S_n(f)(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_n(t) dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} (f(t) - \operatorname{sgn} D_n(t)) D_n(t) dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(D_n(t)) \cdot D_n(t) dt$

Since $\left| \frac{1}{\pi}\int_{-\pi}^{\pi} (f(t) - \operatorname{sgn} D_n(t)) \cdot D_n(t) dt \right| \le \|D_n(t)\|_{\infty} \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} |f(t) - \operatorname{sgn} D_n(t)| dt$

$\le (n + \frac{1}{2}) \cdot \frac{\varepsilon}{n+1} < \varepsilon$

$\Rightarrow S_n(f)(0) \ge \lambda_n - \varepsilon$

[9] Prove that $\|\sigma_n(f)\|_2 \le \|f\|_2$ and $\|\sigma_n(f)\|_\infty \le \|f\|_\infty$

pf: ① Since $\sigma_n(f) = \frac{1}{n}\sum_{j=0}^{n-1} S_j(f)$,

$$\Rightarrow \|\sigma_n(f)\|_{L^2} \le \frac{1}{n}\sum_{j=0}^{n-1} \|S_j(f)\|_{L^2} \le \frac{1}{n}\cdot n\|f\|_{L^2}.$$

②

$$|\sigma_n(f)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+t)|\cdot|K_n(t)|\,dt$$

$$\overset{K_n(t)\ge 0}{\le} \|f\|_{L^\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = \|f\|_{L^\infty}.$$

$$\Rightarrow \|\sigma_n(f)\|_{L^\infty} \le \|f\|_{L^\infty}$$

[10] (a) If $f, K \in C^{2\pi}$, prove $g(x) = \int_{-\pi}^{\pi} f(x+t)K(t)\,dt$ is in $C^{2\pi}$.

(b) If f is 2π-periodic and Riem integrable but $K \in C^{2\pi}$.

is $g(x)$ cts ?

(c) If f, K are 2π-periodic and Riem integrable. Is $g(x)$ still cts ?

pf: (a). Since f is uniformly cts on $[-\pi, \pi]$, choose δ s.t

$$|f(x+t) - f(x+t+\delta)| < \frac{\varepsilon}{2\pi\|K\|_\infty} \quad \text{for all } x, t.$$

Hence, $|g(x) - g(x+\delta)| \le \int_{-\pi}^{\pi} |f(x+t) - f(x+t+\delta)|\,dt \cdot \|K\|_\infty$

$< \varepsilon.$

$\Rightarrow g(x)$ is cts.

Furthermore. g is 2π-peoriodic. Since f is periodic.

(b) Follows from (c),

$\boxed{10\ (c)}$ NTS g is cts

$|g(x)-g(x+\delta)| \le \int_{-\pi}^{\pi} |f(x+t)-f(x+t+\delta)|\, dt \cdot \|K\|_\infty$

By Problem 6, we can choose δ sufficiently small s.t $\int_{-\pi}^{\pi} |f(x+t)-f(x+t+\delta)|^2 dt < \varepsilon$

$\Rightarrow \int_{-\pi}^{\pi} |f(x+t)-f(x+t+\delta)|\, dt \le \sqrt{\varepsilon}\cdot\sqrt{2\pi}$

$\Rightarrow |g(x)-g(x+\delta)| \le \sqrt{\varepsilon}\cdot\sqrt{2\pi}\cdot\|K\|_\infty \Rightarrow g$ is cts.