HW3 chap 14 : 29. 50. 51. 53 chap 16 : 9. 10. 16.

29 Pf: $|S_\alpha(f,P,T)| = |\sum_{i=1}^{n} f(t_i)[\alpha(x_i)-\alpha(x_{i-1})]|$

$$\leq \sum_{i=1}^{n} \|f\|_\infty |\alpha(x_i)-\alpha(x_{i-1})| = \|f\|_\infty V(\alpha,P)$$

50 Proved in problem 26 in HW 2

51 Since f cts, by Corollary 14.21. Let $F(x)=\int_a^x f(t)dt$

$\Rightarrow F'(x)=f(x)=0$ i.e. $f\equiv 0$. on $[a,b]$

53 Construct $\tilde{\alpha}(x)=\alpha(x+)$ and $\tilde{\alpha}(a)=\alpha(a)$, $\tilde{\alpha}(b)=\alpha(b)$.

Then $\tilde{\alpha}$ is right cts and $\tilde{\alpha}$ is bdded variation.

NTS : $\int f d\alpha = \int f d\tilde{\alpha}$, Here, by IBP

$\int_a^b f\, d(\alpha-\tilde{\alpha}) = \int_a^b (\tilde{\alpha}-\alpha) df$

Furthermore, $\tilde{\alpha}-\alpha$ has at most countably discontinuities.

Since f is cts. $\Rightarrow \int_a^b (\tilde{\alpha}-\alpha) df = 0$

$\Rightarrow \int f d\alpha = \int f d\tilde{\alpha}$

Define $\beta(x)=\tilde{\alpha}(x)-\alpha(a)$. then β satisfies the requirement.

To prove uniqueness: if there exists another $\tilde{\beta}$ satisfying all the requirement, then. $0=\int_a^b f(d\beta-d\tilde{\beta})$

for $\forall c\in(a,b)$. construct

$$f(x)=\begin{cases} 1 & , x\in[a,c] \\ 1-\frac{x-c}{\delta} & , x\in(c,c+\delta] \\ 0 & x\in(c+\delta,b] \end{cases}$$ here $c+\delta<b$

Then $\quad 0 = \int_a^b f \, d(\beta - \tilde{\beta}) = (\beta - \tilde{\beta})(c) + \int_c^{c+\delta} f \, d(\beta - \tilde{\beta})$

$\Rightarrow |(\beta - \tilde{\beta})(c)| = \left| \int_c^{c+\delta} f \, d(\beta - \tilde{\beta}) \right| \leq V_c^{c+\delta}(\beta - \tilde{\beta}) \to 0$

This follows from $\quad V_c^{x}(\beta - \tilde{\beta})$ is right cts since $\beta - \tilde{\beta}$ is right cts

$\Rightarrow (\beta - \tilde{\beta})(c) = 0, \quad \forall c \in (a, b)$

Moreover $(\beta - \tilde{\beta})(a) = 0$.

Let $f \equiv 1$ then $\quad 0 = (\beta - \tilde{\beta})(b)$

$\Rightarrow \beta = \tilde{\beta}$

$\boxed{\text{Chap 16.9}}$ $\cdot$ ① $\quad m^*(E) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} l(I_n)$

(sub-additivity)

② $\quad \exists \ E \subset \bigcup_{k=1}^{\infty} J_k$ s.t $\quad m^*(E) + \varepsilon \geq \sum_{k=1}^{\infty} l(J_k)$

Since $\{I_n\}$ are pairwise disjoint.

$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \leq \sum_{k=1}^{\infty} l(J_k) \leq m^*(E) + \varepsilon$

$\Rightarrow \quad m^*(E) = \sum_{n=1}^{\infty} l(I_n)$

$\boxed{16.10}$ for each $U_n = \bigcup_{k=1}^{\infty} I_{n,k}$, Here $\{I_{n,k}\}_{k=1}^{\infty}$ are pairwise disjoint open intervals.

By 16.9, we have

$$m^*\left(\bigcup_{k=1}^{\infty} I_{n,k}\right) = \sum_{k=1}^{\infty} l(I_{n,k})$$

Hence, $m^*\left(\bigcup_{n=1}^{\infty} U_n\right) = m^*\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} I_{n,k}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} l(I_{n,k})$

$$= \sum_{n=1}^{\infty} m^*(U_n)$$

$\boxed{16.16}$ ① $m^*(A) \leq m^*(E \cup A)$ since $A \subseteq E \cup A$

② $m^*(E \cup A) \leq m^*(E) + m^*(A) = m^*(A)$.

$\Rightarrow m^*(A) = m^*(E \cup A)$

Similarly $m^*((A \setminus E) \cup E) = m^*(A \setminus E)$

$\Rightarrow m^*(A) = m^*(A \setminus E)$