hw 4 17, 28, 39, 40, 42ab, 53 (i)(iii), 59

$\boxed{17}$: Recall:

E^c Dense $\Leftrightarrow$ $\forall x \in [a,b]$, $\forall \delta > 0$, $\exists b \in E^c$ s.t $b \in (x-\delta, x+\delta)$

Since E measurable, E^c measurable.

Since $m^*(E) = 0 = m^*(E \cap (x-\delta, x+\delta))$, by 16.16, we have.

$m^*(E^c \cap (x-\delta, x+\delta)) = m^*([a,b] \cap (x-\delta, x+\delta)) = 2\delta > 0$.

Then $\forall x \in [a,b]$. $\forall \delta > 0$, we can find $b \in E^c \cap (x-\delta, x+\delta)$.

Hence, E^c is dense.

$\boxed{28}$ ① let C_i denote the i-th stage to the Cantor set

$\Rightarrow m(C_i) = 1 - \sum_{1}^{i} 2^{n-1}(1-\alpha)3^{-n} = 1 - \frac{1}{2}(\alpha - 1)\sum_{n=1}^{i}\left(\frac{2}{3}\right)^n$.

$\Delta_\alpha \subset C_i$

$\Rightarrow m(\Delta_\alpha) \le \lim_{i\to\infty} m(C_i) = \alpha$.

② Moreover, $m(\Delta_\alpha^c) = \sum_{n=1}^{\infty}(1-\alpha)\frac{2^{n-1}}{3^n} = 1-\alpha$, since the removed intervals are disjoint.

③ $1 \geq m(\Delta_\alpha) + m(\Delta_\alpha^c) \geq m^*([0,1]) = 1$

$\Rightarrow m^*(\Delta_\alpha) = \alpha$.

39 Both A and B are measurable, then. $A \setminus B$ is measurable

$$A = (A \setminus B) \cup B \quad , \quad A \setminus B \cap B = \phi$$

$$\Rightarrow \quad m(A) = m(A \setminus B) + m(B)$$

40 Since A, B measurable, also A and $B \setminus A$ are disjoint, then

$$m(A \cup B) = m(A) + m(B \setminus A)$$

Furthermore, $m(B) = m(B \cap A^c) + m(A \cap B)$ since $B \cap A^c$ and $B \cap A$ are disjoint

$$\Rightarrow \quad m(A \cup B) + m(A \cap B)$$

$$= m(A) + m(B \cap A^c) + m(A \cap B) = \quad m(A) + m(B)$$

42 (a) Defn $f(x) = m(E \cap (-\infty, x])$

Assume $y > x$. then

$$|m(E \cap (-\infty, y]) - m(E \cap (-\infty, x])| \leq m(E \cap (x, y]| \leq |x - y|$$

i.e. $|f(x) - f(y)| \leq |x - y|$, f is Lipschitz cts.

$$\lim_{x \to +\infty} f(x) = m(E) = 1$$

$$\lim_{x \to -\infty} f(x) = 0.$$

By Intermediate Value theorem. $\exists \, x_0$ s.t $f(x_0) = m(E \cap (-\infty, x]) = \frac{1}{2}$.

(b). $\mathbb{Q}$: all the rational numbers in $\mathbb{R}$

Then $m(E) = m(E \cap \mathbb{Q}) + m(E \cap \mathbb{Q}^c) = 1$

Since $m(E \cap \mathbb{Q}) = 0 \Rightarrow m(E \cap \mathbb{Q}^c) = 1$.

By inner. regularity of Lebesgue measure, we can find. a.

closed set. K s.t $K \subset E \cap \mathbb{Q}^c$ and $m(K) > \frac{1}{2}$

Now Define $g(x) = m((-\infty, x] \cap K]$. Similarly, we can show g cts.

and $\lim_{x \to +\infty} g(x) > \frac{1}{2}$

$\lim_{x \to -\infty} g(x) = 0$.

Hence, there exist x_0 s.t $g(x_0) = m((-\infty, x_0] \cap K) = \frac{1}{2}$.

Here $(-\infty, x_0] \cap K$ is closed and consist entirely of irrationals.

[53]. (i). Since each open set can be written as countable union of open intervals., then $\mathcal{B}$ is generated by the open intervals set $\Rightarrow \mathcal{B} = \sigma(\mathcal{E}_1)$

(ii) ① $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{(b-a)}{10^n}, b - \frac{(b-a)}{10^n}]$

Hence $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{B} \subseteq \sigma(\mathcal{E}_2)$

② $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$

$\Rightarrow \mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1) \Rightarrow \sigma(\mathcal{E}_2) \subseteq \mathcal{B}$

Hence. $\mathcal{B} = \sigma(\mathcal{E}_2)$

59. $\forall t \in \mathbb{R}$, Let $A_t = \{B \subseteq \mathbb{R} : t+B \in \mathcal{B}\}$

(i) $\phi \in A_t$:

Since $t + \phi = \phi \in \mathcal{B}$

(ii) $B \in A_t$, then $t+B \in \mathcal{B} \Rightarrow t+B^c = (t+B)^c \in \mathcal{B} \Rightarrow B^c \in A_t$

(iii) $\forall n \in \mathbb{N}^+$, $B_n \in A_t$. then for $\forall n$, $t+B_n \in \mathcal{B}$

$\Rightarrow t + \bigcup_n B_n = \bigcup_n (t+B_n) \in \mathcal{B} \Rightarrow \bigcup_n B_n \in A_t$.

(iiii). for all open sets O, $t+O$ is open. then $t+O \in \mathcal{B}$.

$\Rightarrow O \in A_t \Rightarrow \mathcal{B} \subseteq A_t$

$\Rightarrow$ if E is a Borel Set, then $E+x$ is a Borel set.

Similar proof of the case of "rE"