hw 5 Chap 16: 60, 61, 64, 65, 70, Chap 17: 4, 6.

$\boxed{60}$ ① If E is measurable. then for $\forall \varepsilon > 0$. $\exists$ open set $G \supset E$ s.t. $m^*(G \setminus E) < \varepsilon$

Since $m^*(G \setminus E) = m^*(G+x \setminus E+x) < \varepsilon$ $\Rightarrow$ $E+x$ is measurable.

② $m^*(rG \setminus rE) \le |r|\, m^*(G \setminus E) < \varepsilon$ $\overset{\text{Thm 16.21}}{\Rightarrow}$ rE is measurable.

$\boxed{61}$ Let $E_n = [n, \infty)$. $m(E_n) = \infty$ and $E_n \to \phi$

$\boxed{64}$ ($\Rightarrow$) since E is meas., then

$m(E) = \sup \{ m(K) : K \subset E$ and K is compact$\}$

Since $m(E) < \infty$. $\Rightarrow \forall \varepsilon > 0$. $\exists$ compact set $F \subset E$ s.t $m(F) > m(E) - \varepsilon$

($\Leftarrow$). Since $m^*(E) < \infty$. then for $\forall \varepsilon > 0$. $\exists E \subset O$ (O is open)

s.t $m^*(O) \le m^*(E) + \varepsilon$.

Furthermore, there exists compact set $F \subset E$ s.t

$m(F) > m^*(E) - \varepsilon$.

$\Rightarrow m^*(O) - m^*(F) < 2\varepsilon$

By Lemma 16.15 (iii).,

$m^*(O \setminus F) = m^*(O) - m^*(F) < 2\varepsilon$

$\Rightarrow m^*(E \setminus F) < 2\varepsilon$ $\Rightarrow$ E is measurable

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① $m(E \triangle E) = m(\phi) = 0$ (Reflective : $E \sim E$)

② symmetric : ($E \sim F \Rightarrow F \sim E$)

pf: $E \sim F \Rightarrow m(E \triangle F) = 0 = m((E\setminus F) \cup (F \setminus E)) = m(F \triangle E) \Rightarrow F \sim E$

③ Transitivity ($E \sim F$, $F \sim G \Rightarrow E \sim G$)

Since $m((E\setminus F) \cup (F \setminus E)) = 0 \Rightarrow m(E \cap F^c) = 0 = m(F \cap E^c)$

Since $m((F\setminus G) \cup (G\setminus F)) = 0 \Rightarrow m(F \cap G^c) = 0 = m(G \cap F^c)$

$\Rightarrow m(E \cap G^c) = m(E \cap F \cap G^c) + m(E \cap F^c \cap G^c)$

$\leq m(F \cap G^c) + m(E \cap F^c) = 0$

$\Rightarrow m(E \cap G^c) = 0$,

Similarly $m(G \cap E^c) = 0$

$\Rightarrow m(E \triangle G) \leq m(E \cap G^c) + m(G \cap E^c) = 0 \Rightarrow E \sim G$.

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Approach 1: ① Arbitray union of the open interval is open.

② Other intervals can be represented as the basic operations of open intervals and closed intervals. Hence, we consider all the intervals are closed intervals.

③ Let $\mathcal{C}$ be the collection of all closed intervals $J \subset I_\alpha$ s.t $J \subset I_\alpha$ for some α, Then

$$\bigcup_{\alpha \in A} J_\alpha = \underbrace{\bigcup_{\alpha \in A} J_\alpha^\circ}_{\text{interior pt}} \cup \Big(\bigcup_{\alpha \in A} J_\alpha \setminus \underbrace{\bigcup_{\alpha \in A} J_\alpha}_{\text{endpt}} \Big) = E \cup F \cup G$$

E is open, F consists of all left endpts of J_α but not in any of J_α°

G consists of all right endpts of J_α but not in any of J_α°.

Claim: F are consisting of countable points.

Define $f: F \to \mathbb{Q}$ for $x \in F$ Define $f(x) = q \in \mathbb{Q}$.

Here q satisfies $[x, q] \subset J_\alpha$

Here f is injective. Otherwise, $f(x) = f(y) = q$ and $x < y$, then y is interior point, which contradicts to the definition of F.

Similarly, G also consists of countable points.

$\Rightarrow \bigcup_\alpha J_\alpha$ is measurable. $\Rightarrow \bigcup_\alpha I_\alpha$ is measurable.

Approach 2: Let C be the collection of all closed intervals J such that $J \subset I_\alpha$ for some α. Then C forms a Vitali cover for $\bigcup_\alpha I_\alpha$. By Vitali's covering Thm (6.2), we have

$$m\left(\bigcup_\alpha I_\alpha \setminus \bigcup_{n=1}^{\infty} J_n\right) = 0 \qquad J_n \text{ is closed}$$

$\Rightarrow \bigcup_\alpha I_\alpha$ is the union of F_σ set and a zero-measure set

$\Rightarrow \bigcup_\alpha I_\alpha$ is measurable.

$\boxed{17.4}$ $\Rightarrow$ if χ_E is measurable, then $\chi_E^{-1}((0,\infty)) = E \Rightarrow E$ is measurable.

$\Leftarrow$ if E is measurable

$$\chi_E^{-1}((a, +\infty)) = \begin{cases} \mathbb{R} & a < 0 \\ E & 0 \le a < 1 \\ \emptyset & a \geq 1 \end{cases}$$

$\Rightarrow \chi_E$ is measurable.

$\boxed{17.6}$ $\Rightarrow$ By definition

$\Leftarrow$ Suppose $\{f > \alpha\}$ is measurable for each rational α. Then.

for $\forall \beta \in \mathbb{R}$. Let $\lim_{n\to\infty} \alpha_n = \beta$. here $\alpha_n \in \mathbb{Q}$ and $\alpha_n > \beta$

Then $\{f > \beta\} = \bigcup_{n\in\mathbb{N}^+} \{f > \alpha_n\}$ is measurable

$\Rightarrow f$ is measurable.