hw6 : Chap 17. 14. 19. 31. 33. 36

[14] $\mathcal{A} = \{A : f^{-1}(A) \in M\}$

① $f^{-1}(\phi) = \phi \in \mathcal{M} \Rightarrow \phi \in \mathcal{A}$

② if $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{A}$, then

$f^{-1}(A_1 \cup A_2) = f^{-1}(A_1) \cup f^{-1}(A_2) \in \mathcal{M} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$

③ if $A_1 \in \mathcal{A}$. then $f^{-1}(A_1^c) = f^{-1}(A_1)^c \in \mathcal{M} \Rightarrow A_1^c \in \mathcal{A}$

④ A open. $\Rightarrow f^{-1}(A) \in \mathcal{M}$

since f is measurable.

$\Rightarrow$ Borel $\mathcal{B} \subseteq A$

$\Rightarrow$ if $B \in \mathcal{B}$, then $f^{-1}(B) \in \mathcal{M}$

[19] f. g meas $\Rightarrow$ f-g is measurable

$\Rightarrow \{f-g>0\} = \{f>g\}$ is measurable.

[31] Let $C = \{x \in D : \lim_{n\to\infty} f_n(x) \text{ exists}\}$.

Then C is the set where $f_n(x)$ is Cauchy

($\forall n$, $\exists N_n$ s.t for $k, l > N_n$, we have $|f_k(x) - f_l(x)| < \frac{1}{n}$)

$\Rightarrow C = \bigcap_{n=1}^{\infty} \bigcup_{N_n=1}^{\infty} \bigcap_{k>N_n,\, l>N_n} \{x : |f_k(x) - f_l(x)| < \frac{1}{n}\}$

$\Rightarrow$ C is measurable since $\{f_n\}$ is a sequence of measurable functions

[33] ①

$$f'(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} = \lim_{n\to\infty} g_n \quad (\text{everywhere})$$

g_n is cts (⇒ Borel meas) ⇒ $f'(x)$ is Borel measurable.

(or f is differentiable ⇒ f' is cts ⇒ f' is borel measurable)

② $f'(x) = \lim_{n\to\infty} g_n(x)$ a.e.

g_n is lebesgue measurable $\xRightarrow{\text{Cor 17.12}}$ f' is lebesgue measurable.

[36] Since (f_n) converges almost uniformly to f.

Then for each k, choose E_k s.t $m(E_k) < \frac{1}{k}$ and $f_n \rightrightarrows f$ off E_k

① $m(\bigcap_{k=1}^{\infty} E_k) = 0$ since $m(\bigcap_{k=1}^{\infty} E_k) \le m(E_k) < \frac{1}{k}$ for $\forall k \in \mathbb{N}$

② for $x \in (\bigcap_{k=1}^{\infty} E_k)^c$, then there exists k_0 such that

$x \in E_{k_0}^c$ ⇒ $f_n(x) \to f(x)$.

⇒ (f_n) converges to f at $(\bigcap_{k=1}^{\infty} E_k)^c$