HW 9. 23, 24, 27, 29, 43, 47d, 61

Problem 23

If $\{f_n\}$ is a sequence of Lebesgue integrable functions on $[a,b]$ and if $f_n \rightrightarrows f$ on $[a,b]$. prove: f is integrable and $\int_a^b |f_n - f| \to 0$

pf: Since $f_n \rightrightarrows f$, $\exists N$ s.t $\|f_n - f\| \le 1$ if $n \ge N$

For $n \ge N$, $|(f_n - f)(x)| \le 1 \Rightarrow |f(x)| \le |f_n(x)| + 1$, $\Rightarrow f$ integrable on $[a,b]$

Also $|f_n(x)| \le |f(x)| + 1$ for all $n \ge N$

By Dominated convergence theorem $\Rightarrow \int_a^b |f_n - f| \to 0$

Problem 24

Prove that $\int_0^\infty e^{-x} dx = \lim_{n\to\infty} \int_0^n (1 - (x/n))^n dx = 1$

[Hint: For x fixed, $(1-(x/n))^n$ increases to e^{-x} as $n \to \infty$]

pf: We know $(1 - (\frac{x}{n}))^n \nearrow e^{-x}$ as $n \to \infty$

and $\chi_{[0,n)} \nearrow \chi_{[0,+\infty)}$ as $n \to \infty$.

By MCT, considering the sequence $(1 - (\frac{x}{n}))^n \chi_{[0,n)}$

$$\int_0^\infty e^{-x} dx = \lim_{n\to\infty} \int_0^n (1 - (\tfrac{x}{n}))^n dx$$

$$= \lim_{n\to\infty} \frac{n}{n+1} (1 - (\tfrac{x}{n}))^{n+1} \Big|_0^n = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Problem 27

Suppose $E \subset [0, 2\pi]$ is measurable and that $\int_E x^n \cos x \, dx = 0$ for all $n = 0, 1, 2 \ldots$

Show that $m(E) = 0$

pf: From the condition, we have

$$\int_0^{2\pi} p(x) \cos x \cdot \chi_E \, dx = 0 \quad \text{for } p(x) \text{ any polynomial.}$$

Construct $p(x) = (x - \frac{\pi}{2})(x - \frac{3\pi}{2})$, then

$$p(x) \cos x > 0 \quad \text{except for } \{\tfrac{\pi}{2}, \tfrac{3\pi}{2}\}$$

$$\Rightarrow p(x) \cos x \cdot \chi_E = 0 \quad \text{for } [0, 2\pi] \setminus A \quad \text{with } m(A) = 0$$

$$\Rightarrow \chi_E = 0 \quad \text{for } [0, 2\pi] \setminus (A \cup \{\tfrac{\pi}{2}, \tfrac{3\pi}{2}\})$$

$$\Rightarrow E \subset A \cup \{\tfrac{\pi}{2}, \tfrac{3\pi}{2}\}$$

$$\Rightarrow m(E) \leq m(A) + m(\{\tfrac{\pi}{2}, \tfrac{3\pi}{2}\}) = 0 \quad \Rightarrow m(E) = 0.$$

Problem 29:

If f, (f_n) are lebesgue integrable, and if (f_n) increases pointwise to f, does it follow that $\int f_n \to \int f$ Explain?

pf: Define $g_n = f_n - f_1$, then $g_n \geq 0$ and $g_n \uparrow f - f_1$,

Furthermore, since f, (f_n) are Lebesgue integrable, we have

(g_n), $(f - f_1)$ are lebesgue integrable.

By MCT, we have

$$\lim_{n\to\infty} \int g_n = \int f - \int f_1$$

$$\Rightarrow \lim_{n\to\infty} \int f_n - \int f_1 = \int f - \int f_1$$

Since $\int f_1$ is finite $\Rightarrow \lim_{n\to\infty} \int f_n = \int f$

43. Let f meas. finite a.e on $[0,1]$

(a) If $\int_E f = 0$ for all $E \subset [0,1]$ with $m(E) = \frac{1}{2}$, prove $f = 0$ a.e on $[0,1]$

(b) If $f > 0$, show $\inf\{\int_E f : m(E) \geq \frac{1}{2}\} > 0$.

Pf: (a) Define $A = \{f > 0\}$

$B = \{f < 0\}$

$C = \{f = 0\}$.

A, B, C disjoint and $A \cup B \cup C = [0,1]$ $\Rightarrow m(A) + m(B) + m(C) = 1$

Claim: $m(A) < \frac{1}{2}$, $m(B) < \frac{1}{2}$.

By symmetric properties, we prove the claim for set A.

If $m(A) \geq \frac{1}{2}$, then $\exists A' \subset A$ with $m(A') = \frac{1}{2}$, $\Rightarrow \int_{A'} f = 0 \Rightarrow f = 0$ a.e on A'

contradicting to the definition of A

Therefore, $m(A) < \frac{1}{2}$, $m(B) < \frac{1}{2}$

$\Rightarrow m(B \cup C) > \frac{1}{2}$ and $m(A \cup C) > \frac{1}{2}$.

Now we prove $m(A) = 0$. Otherwise, we can find $A' \subset A$, $C' \subset C$, s.t

$m(A' \cup C') = \frac{1}{2}$ and $m(A') > 0$

Then $\int_{A' \cup C'} f = \int_{A'} f = 0 \Rightarrow f = 0$ a.e on A'. Contradiction!

Therefore, $m(A) = 0$. Similarly, $m(B) = 0$ $\Rightarrow m(C) = 1$

$\Rightarrow f = 0$ a.e on $[0,1]$

b) Define $F_n = \{x \in [0,1] : f(x) > \frac{1}{n}\}$ $\Rightarrow \lim_{n \to \infty} m(F_n) = 1$

$\forall \varepsilon$, $\exists N$, s.t if $n \geq N$, $m(F_n) > 1 - \varepsilon$ (i.e. $m([0,1] \setminus F_n) < \varepsilon$).

Since $m(E \cap F_N) = m(E) - m(E \cap F_N^c) \geq m(E) - \varepsilon \geq \frac{1}{2} - \varepsilon$

Choose $\varepsilon = \frac{1}{4}$, for the corresponding N, we have $m(E \cap F_N) \geq \frac{1}{4}$

$\Rightarrow \int_E f \geq \int_{E \cap F_N} f \geq \int_{E \cap F_N} \frac{1}{N} = \frac{1}{N} m(E \cap F_N) \geq \frac{1}{4N} > 0$.

[4] d. : Compute $\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2} dx$

① $a>0$: $0 < \frac{n}{1+n^2x^2} \le \frac{n}{n^2x^2} \le \frac{1}{x^2}$ for $n \ge 1$

$\lim_{n\to\infty} \frac{n}{1+n^2x^2} = 0$, and $\int_a^\infty \frac{1}{x^2} < \infty$

By DCT $\Rightarrow \lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2} dx = 0$

② $a=0$ $\int_0^\infty \frac{n}{1+n^2x^2} dx = \int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}$

$\lim_{n\to\infty} \int_0^\infty \frac{n}{1+n^2x^2} dx = \frac{\pi}{2}$

In this case (Improper Riem Integral coincides with Lebe integral)

③ $a<0$, $\int_a^\infty \frac{n}{1+n^2x^2} dx = \int_0^\infty \frac{n}{1+n^2x^2} dx + \int_a^0 \frac{n}{1+n^2x^2} dx$

$\Rightarrow \lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2} dx = \frac{\pi}{2} + \lim_{n\to\infty} \int_a^0 \frac{n}{1+n^2x^2} dx = \lim_{n\to\infty} -\arctan(an) + \frac{\pi}{2} = \pi$

[6] Given $f \in L_1[0,2\pi]$ and $\varepsilon>0$. show $\exists$ trig polynomial T s.t $\int_0^{2\pi} |f-T| < \varepsilon$,

Pf: By Thm 18.27 (iii), we can find a cts function g with $g(0)=0=g(2\pi)$ such that $\int_0^{2\pi} |f-g| < \varepsilon/2$. By setting $g(x \pm 2n\pi) = g(x)$ for any $n \in \mathbb{N}$.

Now assume that $g \in C^{2\pi}$.

By Weierstrass's second theorem, (Cor 15.8) we can find T s.t $\|g-T\|_\infty < \frac{\varepsilon}{4\pi}$. ($T$ trig poly)

Then. $\int_0^{2\pi} |f-T| \le \int_0^{2\pi} |f-g| + \int_0^{2\pi} |g-T| \le \varepsilon$