Chap 19 : 43. 46. 49. 60.

$\boxed{43}$. If $f \in L_\infty$, is $m\{|f| = \|f\|_\infty\} > 0$? Is $\{|f| = \|f\|_\infty\} \neq \phi$? Explain

Sol : Let $f(x) = \arctan x$, $x \in \mathbb{R}$

$$\|f\|_\infty = \frac{\pi}{2}$$

But $\{|f| = \|f\|_\infty\} = \phi$

$\boxed{46}$ Let $f \in C[0,1]$ and $0 \le A < \infty$. If $|f(x)| \le A$ for a.e. $x \in [0,1]$, prove that $|f(x)| \le A$ for all $x \in [0,1]$. Conclude that

$$\sup_{0 \le x \le 1} |f(x)| = \operatorname{ess.sup}_{0 \le x \le 1} |f(x)|.$$

pf: Assume $|f(x_0)| > A$ for some $x_0 \in [0,1]$

We can assume $|f(x_0)| > A + \varepsilon$ for some small enough ε.

By continuity of f. $\exists\ \delta$. s.t for $\forall\ x \in (x_0 - \delta, x_0 + \delta) \cap [0,1] := F$

$|f(x)| > A + \varepsilon$.

Recall that $\|f\|_\infty = \inf\{M \ge 0 : m\{x \in E : |f| > M\} = 0\}$

If $M \le A + \varepsilon$. then $m\{x \in [0,1] : |f| > M\} \ge m\{x \in [0,1] : |f| > A + \varepsilon\} \ge m(F) > 0$

$\Rightarrow \|f\|_\infty \ge A + \varepsilon$ Contradiction.

Therefore, $\|f\|_{C[0,1]} = \|f\|_{L^\infty[0,1]}$

[49] Prove that $L^\infty(\mathbb{R})$ is not separable. More generally, if $m(E)>0$, then $L^\infty(E)$ is not separable

pf: If $m(E)>0$, then E can be partitioned into a countable union of almost disjoint measurable subsets $A_0, A_1, \dots$ and that all have positive measure.

(Using Exercise 16.42)

Now, for each subset M of $\mathbb{N}$, Define

$$f_M(x) = \begin{cases} 1 & \text{if } x \in A_n \text{ and } n \in M. \\ 0 & \end{cases}$$

Then $\{f_M\}_{M \subseteq \mathbb{N}}$ uncountable and $\|f_{M_1} - f_{M_2}\|_{L^\infty(E)} = 1$ for different M_1 and M_2.

Consider $B_M = B(f_M, \frac{1}{2}) = \{g \in L^\infty(E) : \|f_M - g\|_{L^\infty} < \frac{1}{2}\}$

If $L^\infty(E)$ is separable, then $\exists$ countable set S s.t each B_M contains at least one element of S.

Since $\{B_M\}_{M \subseteq \mathbb{N}}$ are disjoint balls, then S can not be countable.

Contradiction!

Therefore, $L^\infty(E)$ is not separable.

[60] Fix $1 \le p < \infty$, $f \in L_p[a,b]$, and $\varepsilon > 0$. Show that there is an algebraic polynomial Q and a trig polynomial T s.t $\|f-Q\|_p < \varepsilon$ and $\|f-T\|_p < \varepsilon$.

pf: By Thm 19.13, $\exists\, g \in C[a,b]$ s.t $\|f-g\|_p < \frac{\varepsilon}{2}$.

By Cor 15.9, 15.8, $\exists\, Q$ & T s.t $\|g-Q\|_\infty < \frac{\varepsilon}{2(b-a)^{\frac{1}{p}}}$ and $\|g-T\|_\infty < \frac{\varepsilon}{2(b-a)^{\frac{1}{p}}}$

$\Rightarrow \|g-Q\|_p < \frac{\varepsilon}{2}$ and $\|T-g\|_p < \frac{\varepsilon}{2}$

By Minkowski's ineq, we have $\|f-Q\|_p < \varepsilon$ and $\|f-T\|_p < \varepsilon$.