Shock trace prediction by reduced models for stochastic Burgers equation

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Join work with **Nan Chen and Honghu Liu**

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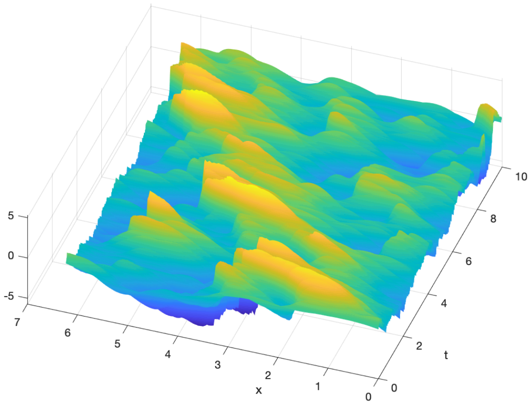
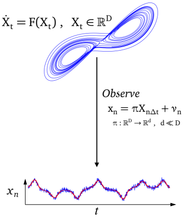
Can a reduced model predict random shocks?



$\nu = 0.05$, $K_0 = 4$ stochastic force $\rightarrow$ random shocks

Prediction with Uncertainty Quantification

$$x' = F(x) + U(x,y), \qquad \text{resolved scales}$$
$$y' = G(x,y), \qquad \text{subgrid-scales}$$
$$\text{Data:}\{x(nh)\}$$



(courtesy of Kevin Lin)

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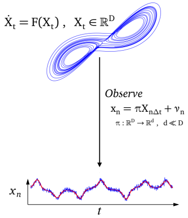

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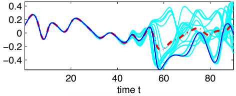

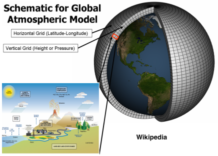

Motivation: <u>Data assimilation:</u>

- ensemble forecasting
- can only afford to resolve $x' = F(x)$

Goal: ensemble prediction of extreme events

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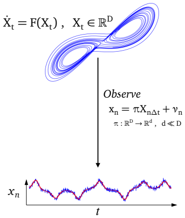

courtesy of Kevin Lin

Objective 1: reduced model $\approx$ the flow map: $x_{1:n-1} \to x_n$

- captures key statistical + dynamical properties
- ensemble simulations (with a larger time-step)

Space-time reduction: spatial dimension $\downarrow$; time-step size $\uparrow$

Objective 2: Predict extreme events

Closure modeling, model error UQ, subgrid parametrization

Direct constructions:

- nonlinear Galerkin [Fioas, Jolly, Kevrekidis, Titi...]
- moment closure [Levermore, Morokoff...]
- Polynomial chaos [Karniadarkis/Najm/Majda/Chorin... groups]
- Mori-Zwanzig formalism
  memory $\rightarrow$ non-Markov process
  [Chorin, Hald, Kupferman, Stinis, Li, Darve, E, Karniadarkis, Venturi, Duraisamy ...]

Data-driven RM

- PCA/POD, DMD, Kooperman [Holmes, Lumley, Marsden, Wilcox, Kutz, Rowley ...]
- ROM closure [Farhat, Carlberg, Iliescu, Wang...]
- stochastic models: SDEs/GLEs, time series models [Chorin/Majda/Gil groups]
- machine learning (...)

- What does a RM approximate?
- RM for random extreme events?

a statistical learning study of RM

Flow map approximation

$x' = F(x) + U(x, y), \; y' = G(x, y).$
Data $\{x(nh)\}_{n=1}^{N}$

Classical numerical schemes

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{F} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

- trajectory-wise Approx.
- Closure flow map (Mori-Zwanzig): $x_n = F_n(x_{1:n-1})$

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Data-driven methods:
$F_n(x_{1:n-1}) \approx \widehat{F}_n(x_{n-p:n-1})$

- average the subgrid-scales
  approximate in distribution
- Learning: curse of dimensionality
  - machine learning: great success
  - parametric inference
    use the structure of the map

$$(X_n - X_{n-1})/h = R_h(X_{n-1}) + \sum_i c_i \phi_i(X_{n-p:n-1}, \xi_{n-p:n-1}) + \xi_i$$

NARMA(p, q) [Chorin-Lu (15)]

$$(X_n - X_{n-1})/h = R_h(X_{n-1}) + \Phi_n + \xi_n,$$

$$\Phi_n = \underbrace{\sum_{j=1}^{p} a_j X_{n-j} + \sum_{j=1}^{r}\sum_{i=1}^{s} b_{i,j} P_i(X_{n-j})}_{\text{Auto-Regression}} + \underbrace{\sum_{j=1}^{q} c_j \xi_{n-j}}_{\text{Moving Average}}$$

- $R_h(X_{n-1})$ from a numerical scheme for $x' \approx F(x)$
- Φ_n depends on the past

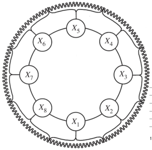
Tasks:

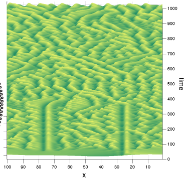
Structure derivation: terms and orders (p, r, s, q) in Φ_n;

Parameter estimation: $a_j, b_{i,j}, c_j$, and σ. Conditional MLE

Chaotic or stochastic systems

- the two-layer Lorenz96 [Chorin-Lu15]
- Kuramoto-Sivashisky [Lu-Lin-Chorin17]
- stochastic Burgers [Lu20]
  - $\nu = 0.05$, $K_0 = 4$ stochastic force
  - Full model: $N = 128, dt = 0.005$
  - Reduced model: $K = 8$, $\delta = 20dt$





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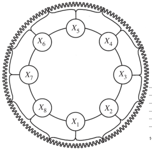
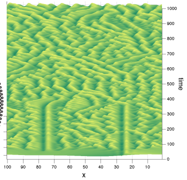
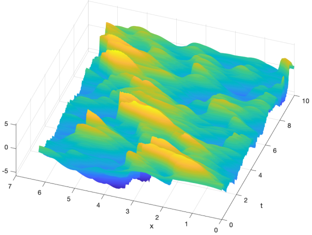




The NARMA model can (for resolved var.)

- tolerate large time-steps
- reproduces statistics: ACF, PDF
- improves Data Assimilation [Lu-Tu-Chorin17]



Prediction of the random shocks?

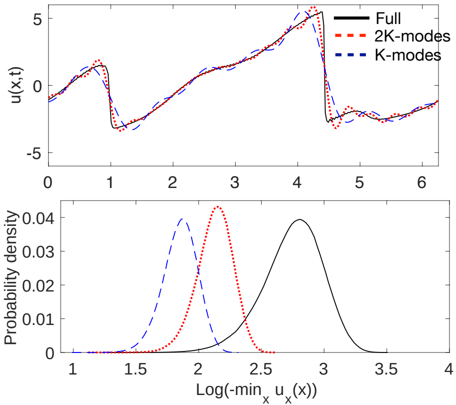
Representing shocks

Representation of shocks

- Full: viscous shocks
- 2K- and K-modes: smooth

Distribution of max derivatives

- Different scales

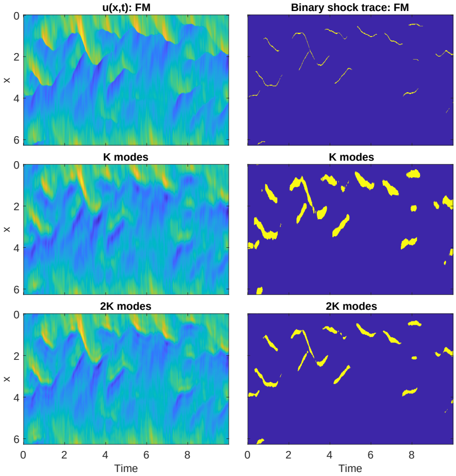


$\Rightarrow$ Shock representation requires high-modes.

Shock trace by thresholding - FM

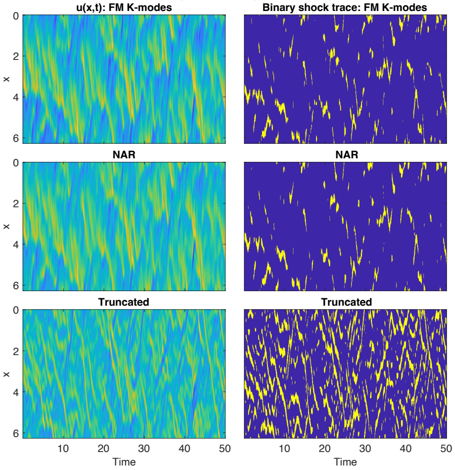
Trace of random shocks

- space–time locations
- resolution-adaptive thresholds
- empirical from FM data



Shock trace prediction with IC + force

Trace in a single trajectory

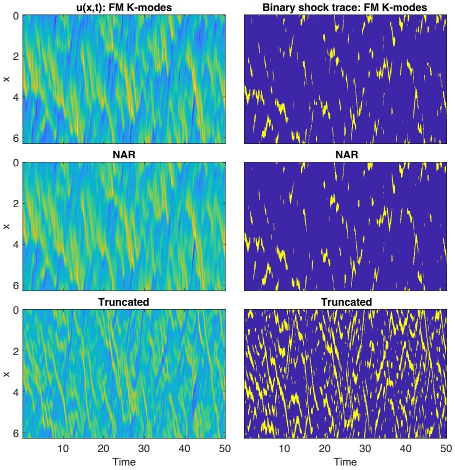


NAR v.s. Truncated

- Significant improvements

Shock trace prediction with IC + force

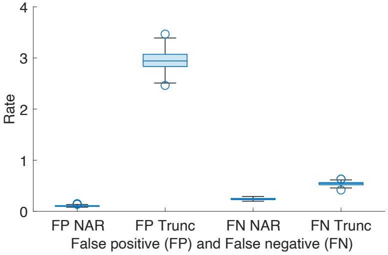
Trace in a single trajectory



NAR v.s. Truncated

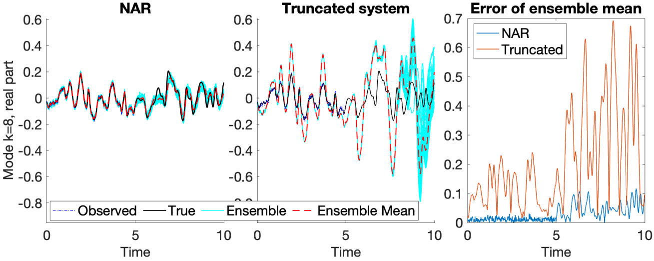
- Significant improvements

Rates of false prediction (200 simuls)



Data assimilation

Ensemble prediction



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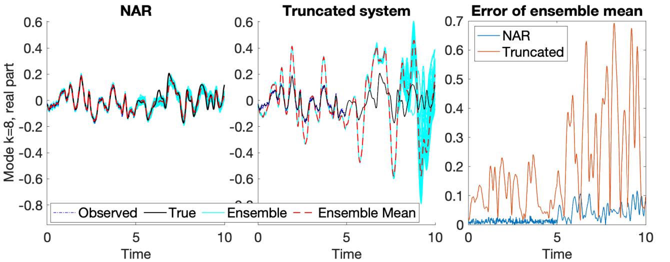
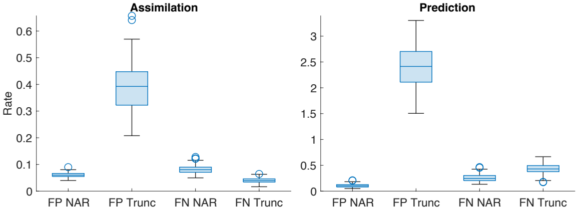
Ensemble prediction



- Assimilation $[0, 5]$, Prediction $(5, 10]$
- Rates from 200 simuls



Summary

$x' = f(x) + U(x,y)$, $y' = g(x,y)$.
Data $\{x(nh)\}_{n=1}^{N}$

↓ **Inference**

"$X_{n+1} = X_n + R_h(X_n) + Z_n$"
for prediction

Numerical + inferential model reduction

- non-intrusive time series (**NARMA**)
- $\approx$ the flow map: $x_{1:n-1} \to x_n$
- space-time reduction

$\to$ Predicts shock trace: space-time locations
(shock representation requires high modes)

Open question: Optimal space-time reduction?

Space-time reduction

- dimension reduction r
- large time-stepping $\delta = Gap * \Delta t$

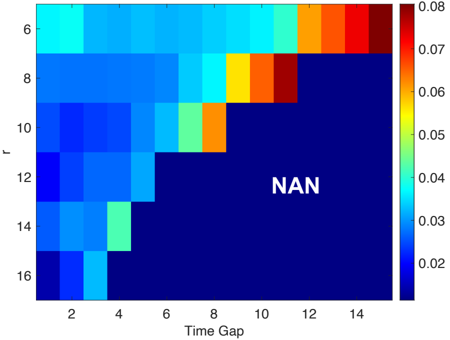
Accuracy of RMs

- 90 RM (r, δ)
- RMSE on of 100 trajs on $[0, 4]$

Observations:

- As $r \uparrow$: accuracy $\uparrow$, tolerate $\delta \downarrow$
- Each r: "sweet spot" medium δ

Trade-off (r, δ) for an "optimal" RM?



References

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- Data assimilation

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Thank you!