1. Area = \( \int_0^\pi \sin(x) \, dx = -\cos(x) \bigg|_0^\pi = -\cos(\pi) + \cos(0) = 2. \)

2. \( y - f(1) = f'(1)(x - 1), \) where \( f(1) = 0 \) and \( f'(1) = -1 \) yields \( y = -x + 1. \)

3. Preliminary observation: \( x \) and \( x - 1 \) must be in the domain of \( \ln, \) therefore \( x > 1. \)

\[ 2 = \ln(x) + \ln(x - 1) = \ln(x(x - 1)) \Rightarrow x(x - 1) = e^2 \Rightarrow x^2 - x - e^2 = 0. \]

The roots of this equation of degree two are \( x = \frac{1 \pm \sqrt{1 + 4e^2}}{2}. \) Of these, only one is greater than one, namely \( x = \frac{1 + \sqrt{1 + 4e^2}}{2}. \)

4. \( \arcsin : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}], \) so we need:

\( e^x \in [-1, 1] \Rightarrow x \leq 0, \) so the domain of \( g \) is \((\infty, 0]. \) Determine the domain of the function \( g(x) = \arcsin(e^x). \)

5. First method. Applying \( \frac{dy}{dx} \) on both sides (implicit differentiation) yields:

\[ 2 \sin(y) \cos(y) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{\sin^2(y)}{\sin(2y)} = \frac{1}{2\sqrt{1-x^2}}. \]

Second method. \( \sin^2(y) = x \Rightarrow \sin(y) = \sqrt{x} \Rightarrow y = \arcsin(x^{1/2}). \) Since the derivative of \( \arcsin(x) \) is \( \frac{1}{\sqrt{1-x^2}}, \) applying chain rule:

\[ \frac{dy}{dx} = \frac{1}{\sqrt{1-x^{1/2}}} (x^{1/2})' = \frac{1}{2} x^{-1/2} \frac{1}{\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}}. \]

6. The idea behind this problem was to see how you would avoid doing some heavy computations.

Logarithmic derivative (meaning: differentiate \( \ln(F(x)) \)) gives:

\[ \ln(F(x)) = \frac{1}{2} \ln(x^2 + 3) + \frac{1}{2} \ln(x + 1) - 2 \ln(x) \]

\[ \frac{F'(x)}{F(x)} = \frac{x}{x^2 + 3} + \frac{3}{2(x+1)} - \frac{2}{x}. \]

\( @ x = 1: \frac{F'(1)}{F(1)} = \frac{1}{2} + \frac{3}{4} - 2 = -1 \Rightarrow F'(1) = -F(1). \)

But \( F(1) = \sqrt{\frac{4}{\sqrt{7}}} = 4\sqrt{2}. \) Therefore \( F'(1) = -4\sqrt{2}. \)

Second method. If you don’t like logarithmic derivatives, there are many ways in which you can simplify a straightforward calculation. One student did this: let

\[ F(x) = \frac{\sqrt{g(x)}}{x^2}. \]

Then \( F'(x) = \frac{\frac{g'(x)}{2x^2\sqrt{g(x)}} - \frac{2\sqrt{g(x)}}{x^3}}{x^2} \Rightarrow F'(x) = \frac{g'(1)}{2\sqrt{g(1)}} - 2\sqrt{g(1)}. \) It is easy to compute \( g(1) = (1 + 3)(1 + 1)^3 = 32 \) and \( g'(x) = 2x(x + 1)^3 + 3(x + 1)^2(x^2 + 3) \Rightarrow g'(1) = 16 + 48 = 64. \)

Hence \( F'(1) = \frac{64}{2\sqrt{32}} - 2\sqrt{32} = \frac{\sqrt{32}}{2} - 2\sqrt{32} = -\sqrt{32} = -4\sqrt{2}. \)

7. a) Correction.

Definition of derivative at 1 for \( f(x) = 2^x: \lim_{x \to 1} \frac{2^x - 2}{x - 1} = f'(1) = 2 \ln(2). \)

However \( \lim_{x \to 1} \frac{2^x - 1}{x - 1} \) doesn’t exist, since numerator \( \to 1 \) while denominator \( \to 0. \)

Date: December 13, 2007.
b) \( \lim_{x \to +\infty} [\ln(x + 5) - \ln(x)] = \lim_{x \to +\infty} \ln \left( \frac{x + 5}{x} \right) = \lim_{x \to +\infty} \ln(1 + \frac{5}{x}) = \ln(1) = 0 \).

8. a) Applying L'Hospital rule (case \( \frac{0}{0} \) yields):
\[
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.
\]
Determine the following limits by any method:

b) There are many way of doing this. Using a substitution \( x = \frac{1}{y} \) gives:
\[
\lim_{y \to 0^+} \sqrt{y} \ln(y) = \lim_{y \to +\infty} \frac{\ln\left(\frac{1}{y}\right)}{y} = \lim_{y \to +\infty} -\frac{2\ln(y)}{y} = 0 \text{ after applying L'Hospital once.}
\]

Alternative method: Applying L'Hospital in the \( \frac{\infty}{\infty} \) case yields
\[
\lim_{x \to 0^+} \frac{\ln(x)}{x^{1/2}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{1}{2}x^{-3/2}} = \lim_{x \to 0^+} -2x^{1/2} = 0.
\]

9. a) \( f(x) = (x^2 + 2x)e^{-x} \Rightarrow f'(x) = (2x + 2)e^{-x} - (x^2 + 2x)e^{-x} = (2 - x^2)e^{-x} \).
\( f'(x) = 0 \Rightarrow 2 - x^2 = 0 \Rightarrow x = \pm \sqrt{2} \). These are the critical points.

b) Differentiate \( f'(x) \) to obtain: \( f''(x) = -2xe^{-x} - (2 - x^2)e^{-x} = (x^2 - 2x - 2)e^{-x} \).

Second derivative test:
\( f''(\sqrt{2}) = -2\sqrt{2}e^{-\sqrt{2}} < 0 \Rightarrow \sqrt{2} \) is a local min.
\( f''(-\sqrt{2}) = 2\sqrt{2}e^{\sqrt{2}} > 0 \Rightarrow -\sqrt{2} \) is a local max.

10. \( h'(x) = \ln(x) + 1 \) and \( h''(x) = \frac{1}{x} \).

Critical points: \( h'(x) = 0 \Rightarrow \ln(x) = -1 \Rightarrow x = e^{-1} \).

On \((0, e^{-1})\): \( h'(x) < 0 \), hence \( h \) is decreasing.

On \((e^{-1}, +\infty)\): \( h'(x) > 0 \), hence \( h \) is increasing.

\( h''(x) > 0 \) everywhere so the graph is concave up.

(Comment. A number of students concluded from the fact that \( h''(x) = 0 \) has no solutions the fact that ‘there is no concavity’, or that the second derivative does not tell us anything about the concavity. It is the sign of the second derivative that dictates the concavity of the function. The analogue of ‘critical point’ would be an ‘inflection point’, but I never covered such things. Same with the monotonicity: this is determined by the sign of \( h' \). I think the confusion stems from certain ‘rules of thumb’ that some of you picked on previous calculus classes and didn’t try to ‘unlearn’.)

\( h(x) = 0 \Rightarrow x \ln(x) = 0 \Rightarrow \ln(x) = 0 \Rightarrow x = 1 \) so the graph intersects the \( x - \text{axis} \) at \( x = 1 \).

Behavior at endpoints:
\[
\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} x \ln(x) = 0.
\]
\[
\lim_{x \to +\infty} h(x) = \lim_{x \to +\infty} x \ln(x) = +\infty.
\]

11. a) \[
\int_1^3 2^x \, dx = \left. \frac{2^x}{\ln(2)} \right|_1^3 = \frac{2^3 - 2^1}{\ln(2)} = \frac{6}{\ln(2)}.
\]

b) \[
\int_1^2 4 + u^2 \, du = \int_1^2 4u^{-3} \, du + \int_1^2 u^{-1} \, du = [-2u^{-2}]_1^2 + \ln(u)|_1^2 = -2(2^{-2} - 1^{-2}) + \ln(2) = \frac{3}{2} + \ln(2).
\]
12. Set $1 - x = y$, then $dx = -dy$ and

\[ \int \frac{x^2}{\sqrt{1-x}} \, dx = - \int \frac{(1-y)^2}{\sqrt{y}} \, dy = - \int \frac{1-2y+y^2}{\sqrt{y}} \, dy \]

\[ = - \int \left( y^{-1/2} - 2y^{1/2} + y^{3/2} \right) \, dy \]

\[ = -y^{1/2} + 2y^{3/2} - \frac{y^{5/2}}{5/2} + C = -2y^{1/2} + \frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2} + C \]

\[ = -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C \]