Homework 3: due Oct 1

Part I

A1. Prove from scratch (using the definitions introduced in class) that \( \sin\left(\frac{\pi}{2} - x\right) = \cos(x) \). Explain how would you use this to draw the graph of \( \cos(x) \) from that of \( \sin(x) \).

A2. a. Use the trigonometric formula 17.b [in Appendix A, page A7] to compute the limits

\[
\lim_{x \to 0} \frac{\cos(x) - 1}{x} \quad \text{and} \quad \lim_{x \to 0} \frac{\cos(x) - 1}{x^2}.
\]

(You may compare your answer with Example 11 of p. 43, but your proof should be different.)
b. Approximate the function \( \cos(x) \) by a quadratic function near \( x = 0 \). [use part a.]
c. Sketch the graph of this quadratic function against the graph of \( \cos(x) \). [It should be a parabola that is tangent to the graph of \( \cos(x) \) at \( x = 0 \).]

A3. Determine the domain of the function \( g(x) = \begin{cases} \sin x & x \neq 0 \\ 0 & x = 0 \end{cases} \).

A4. Determine the domain of the function \( h(x) = \frac{x^2 - 1}{x - 1} \). Determine all the points where \( h \) is continuous. How would you modify \( h \) to make it continuous everywhere?

B1. Give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(0) = 2 \) and \( \lim_{x\to 0} f(x) = +\infty \).

B2. Give an example of a function \( g : (0, \infty) \to \mathbb{R} \) such that \( \lim_{x\to a^+} g(x) = +\infty \) and yet \( g \) is not monotone increasing on any interval of the form \((0, a)\).

B3. [If true, argue why. If false, give a precise example that contradicts the statement.]
a. True or false: "if \( \lim_{x\to 1} f(x) = 0 \), then \( \lim_{x\to 1} \frac{1}{f(x)} = +\infty \)."
b. True or false: "if \( \lim_{x\to 1} f(x) = -\infty \), then \( \lim_{x\to 1} \frac{1}{f(x)} = 0 \)."
c. True or false: "if \( \lim_{x\to 3^+} f(x) = +\infty \), then \( f \) is not continuous at 3".

C1. Find \( \lim_{x \to 0} \frac{x^2 \sin^2(x) + x^3}{x} \).

C2. Using the definitions:
a) For \( f(x) = x^2 \), find \( f'(2) \).
b) For \( g(x) = \sqrt{x} \), find \( g'(4) \).
c) For \( h(x) = |x| \), find \( h'(0) \).

C3. Let \( F(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \).
a) Argue that \( F \) is continuous.
b) Determine \( F'(0) \).

Part II. From the textbook:

Section 1.5, p. 55: 4, 10, 15.
Section 1.5, p. 55: 35, 39, 42*(optional), 47
Section 1.6, p.67: 13