1. [Exercise 18, 3.2] Let \( f(x) = 2x - 1 - \sin(x) \). Since \( f(0) = -1 \) and \( f(\pi) = 2\pi - 1 > 0 \), there exists \( x_0 \in (0, \pi) \) such that \( f(x_0) = 0 \). So the equation has at least one root.
Assume the equation \( f(x) = 0 \) has two roots, say \( x_1, x_2 \), labelled such that \( x_1 < x_2 \). By Rolle’s theorem, there exists \( c \in (x_1, x_2) \) such that \( f'(c) = 0 \). But \( f'(x) = 2 - \cos(x) \), so the equation \( f'(c) = 0 \) has no solution, contradiction. Therefore \( f(x) = 0 \) has exactly one root.

2. [Exercise 42, section 3.1] \( f'(x) = \frac{4-x^2}{(x^2+4)^2} \). Critical points: \( f'(x) = 0 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2 \).

3. [Exercise 44, section 3.1] \( g'(x) = 1 - 2\sin(x) = 0 \Rightarrow \sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \).
\( g''(x) = -2\cos(x) \).
\( g''(\frac{\pi}{6}) = -2\cos(\frac{\pi}{6}) = -\sqrt{3} \Rightarrow x = \frac{\pi}{6} \) is a point of local maximum.
\( g''(\frac{5\pi}{6}) = -2\cos(\frac{5\pi}{6}) = \sqrt{3} \Rightarrow x = \frac{5\pi}{6} \) is a point of local minimum.

4. [Exercise 25, section 3.2] Assume such a function exists. MVT \( \Rightarrow \exists c \in (0, 2) \) such that \( f'(c) = \frac{f(2) - f(0)}{2-0} = \frac{5}{2} \), which contradicts the fact that \( f'(x) \leq 2 \), for all \( x \).
Therefore our assumption is wrong, such a function does not exist.

5. \( \int_0^2 x(x-2)dx = -\text{Area} \).
\( \int_0^2 (x^2 - 2x)dx = \int_0^2 x^2dx - 2 \int_0^2 xdx = \left[ \frac{x^3}{3} \right]_0^2 - \left[ \frac{x^2}{2} \right]_0^2 = -\frac{4}{3} \).
Therefore \( \text{Area} = \frac{4}{3} \).

6. With the substitution: \( x^2 + 1 = y, 2xdx = dy \), we have
\( \int \frac{xdx}{(x^2+1)^{3/2}} = \int \frac{dy}{y^{3/2}} = -\frac{1}{2}y^{-1/2} + C = -\frac{1}{2}(x^2+1)^{-1/2} + C \).
Hence \( \int_0^1 \frac{xdx}{(x^2+1)^{3/2}} = \left[ -\frac{1}{2}(x^2+1)^{-1/2} \right]_0^1 = \left( -\frac{1}{4} \right) - \left( -\frac{1}{2} \right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \).

7. [Exercise 51, section 4.3 (modified)]
\( \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n} = \frac{1}{n} \sqrt{\frac{1}{n} + \frac{1}{n} + \frac{2}{n} + \ldots + \frac{n}{n}} = R_n \) the \( n^{th} \) Riemann sum for \( f(x) = \sqrt{x} \) over \( [0, 1] \).
Therefore \( \lim \text{lim}_n \to \infty R_n = \int_0^1 \sqrt{x}dx = \left[ \frac{2}{3}x^{3/2} \right]_{x=0}^{x=1} = \frac{2}{3} \).

Date: November 14, 2007.