1. This exam has 7 pages including this cover.
2. Books, notes or calculators are not allowed.
3. The correct answer without appropriate justification is worth zero points. For full credit we must be able to see how you got your answer.
Part I
In Part I and II the function $\Phi : (r, \theta, t) \in [0, 1] \times [0, 2\pi] \times [1, \infty) \to \mathbb{R}^3$ defined by
$$\Phi(r, \theta, t) = (r \cos \theta, r \sin \theta, tr)$$
parametrizes a solid cone centered around the $z$-axis, with vertex at the origin; it is the solid of revolution obtained by rotating the line $y = z$ ($y \geq 0$) around the $z$-axis.

Let $C$ be the part of this cone below the plane $z = 1$: it is parametrized by the region where $1 \leq t \leq 1/r$.

Its surface boundary $\partial C$ has two parts: the top disk $D$ is parametrized by $(x, y, z) = \Phi(r, \theta, 1/r)$ and the remaining (lateral) surface $\Sigma$ is parametrized by $(x, y, z) = \Phi(r, \theta, 1)$. The unit normal vector to $\partial C$ is pointing outward.

Problem 1. Calculate the Jacobian determinant for $\Phi$, and use that to calculate the volume of the cone $C$ (change of variable formula).
**Problem 2.** Compute the volume of $C$ by any other means. (Compare your answers.)

**Problem 3.** Use Gauss’ theorem to calculate the flux integral $\int \int_{\Sigma} \vec{r} \cdot d\vec{S}$ where $\vec{r}$ is the position vector field.
Part II

Problem 4. Calculate (directly) the flux of the vector field

\[ \vec{F}(x, y, z) = \frac{zi - xk}{x^2 + z^2} \]

through \( \Sigma \) when the unit normal vector \( \vec{n}_\Sigma \) is pointing outside (i.e. it has negative \( k \) component).

Hint: The formula \( \int \frac{du}{a^2 - u^2} = \frac{1}{2a} \log \left| \frac{a + u}{a - u} \right| \) may be useful.
Problem 5. Use Stokes’ theorem (and the results of the previous problem) to evaluate the integral of the vector field

$$\overrightarrow{G}(x, y, z) = (yx^{-1}, -\frac{1}{2} \log(1 + x^{-2}z^2), 0)$$

around the circular loop $t \mapsto (\cos t, \sin t, 1)$ where $0 \leq t \leq 2\pi$. 
Part III

Problem 6. Calculate the area of the portion of the unit sphere \( x^2 + y^2 + z^2 = 1 \) where \( z \geq \frac{1}{2} \).

Problem 7. True or false: there exists a vector field \( \vec{F} \) such that \( \nabla \times \vec{F} = \vec{r} \). Justify your answer.
Problem 8. Let $\overrightarrow{\omega} = i + j + k$ and $g(x, y, z) = x^2 e^{z^3 \sin y} + 332x^{10} y^2 z^5$ a scalar function (the actual formula of $g$ is irrelevant). Consider the vector field $\overrightarrow{H} = \overrightarrow{\omega} \times \nabla g$. Find a vector field $\overrightarrow{J}$ such that $\nabla \times \overrightarrow{J} = \overrightarrow{H}$.

Problem 9. Let $S$ the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ defined by $x + y + z \geq 1$. Compute the flux of $\overrightarrow{H}$ through $S$ using Stokes’ theorem (do not use Gauss’ theorem).

**Hint:** What’s the relation between the vector field $\overrightarrow{J}$ and the plane containing $\partial S$?