

WEEK 1: CARDINAL NUMBERS

1. FINITE SETS

- 1.1. For a finite set A , we will denote by $\#A$ the number of elements of A .
- 1.2. If A, B are two finite sets then $\#(A \times B) = \#A\#B$.
- 1.3. $P(A)$, the power set of A , has $2^{\#A}$ elements. To see this, assume $A = \{1, 2, \dots, n\}$. Attach to each subset $S \subset A$ the binary string $y = y_1y_2\dots y_n$ where $y_i = 1$ if $i \in S$ and 0 otherwise. Such a correspondence is bijective, and the number of such binary strings is clearly 2^n . For this reason we sometimes use the notation 2^A instead of $P(A)$. This notation is extended to infinite sets too.
- 1.4. Another way of thinking of $P(A)$ is in terms of functions $f : A \rightarrow \{0, 1\}$. To each subset $S \subset A$ corresponds the function $f : A \rightarrow \{0, 1\}$ given by $f_S(x) = 1$ if $x \in S$ and 0 otherwise.
- 1.5. In general, given two sets A and B , we will denote by $Fun(A, B)$ the set of functions $f : A \rightarrow B$. If A and B are finite, this reason. Note that $\{1, 2\}^A$ is the same with 2^A from above.

2. ARBITRARY SETS

- 2.1. If A and B are two sets, not necessary infinite, we say that $A \sim B$ if there exists a bijective function $f : A \rightarrow B$. This establish an equivalence relation between sets, and we will denote by $|A|$ the equivalence class of a given set A , or shortly its cardinality.
- 2.2. Definition: we say that $|A| \leq |B|$ if there exists an injective function $f : A \rightarrow B$. This is equivalent with the existence of a surjective function $g : B \rightarrow A$.
- 2.3. Theorem (Cantor-Bernstein-Schroeder). For any given sets A and B , if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.
- 2.4. Check that if A and B are finite sets, then $|A| = |B|$ (or $|A| \leq |B|$) if and only if $\#A \leq \#B$ (or $\#A = \#B$).
- 2.5. Notation: from now on we will denote by $|A| \times |B|$, $2^{|A|}$, etc... the cardinality of $A \times B$, $P(A)$, etc..
- 2.6. Cantor's theorem: $2^{|A|} > |A|$, for any given set A .

3. COUNTABLE SETS

- 3.1. Definition: $\aleph_0 = |\mathbb{N}|$. An infinite set A is countable if $|A| = \aleph_0$, in other words, if there exists a bijection $f : A \rightarrow \mathbb{N}$.
- 3.2. $\aleph_0^2 = \aleph_0$, meaning that if A, B are countable then $A \times B$ is countable. This is almost the same as saying that if A_n is countable for $n \geq 1$, then $\cup_{n \geq 1} A_n$ is countable.

4. THE CARDINALITY OF \mathbb{R}

4.1. Looking at the subset of elements which have only 0 and 1 in their decimal expansion, we see that $2^{\aleph_0} \leq |[0, 1]|$. Since the decimal expansion is really a surjective (but not bijective!) function $Fun(\mathbb{N}, \{0, 1, 2, \dots, 9\}) \rightarrow [0, 1]$ (check!), we see that $|[0, 1]| \leq 10^{\aleph_0}$. Therefore $2^{\aleph_0} \leq |[0, 1]| \leq 10^{\aleph_0} \leq 2^{4\aleph_0} = 2^{\aleph_0}$, hence $[0, 1] = 2^{\aleph_0}$.

4.2. Since \mathbb{R} is a countable union of disjoint intervals of length one, $2^{\aleph_0} |\mathbb{R}| = \aleph_0 \times 2^{\aleph_0} \leq 2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$, hence $|\mathbb{R}| = 2^{\aleph_0}$.

4.3. Notation: $\mathfrak{c} = 2^{\aleph_0} > \aleph_0$.

4.4. Prove that $\mathfrak{c}^2 = \mathfrak{c}$.