CAUCHY SEQUENCES OF RATIONAL NUMBERS

1. Definitions

1.1. Definition. A sequence $\{x_n\}_{n\geq 1}$ of rational numbers is said to be Cauchy if it satisfies the following property:

$$\forall \epsilon > 0, \quad \exists N > 0 \quad \text{such that}$$
$$m, n \geq N \Rightarrow |x_m - x_n| \leq \epsilon$$

1.2. Definition. We say that $x_n \to 0$ if it has the following property:

$$\forall \epsilon > 0, \quad \exists N > 0 \quad \text{such that}$$
$$n \geq N \Rightarrow |x_n| \leq \epsilon$$

1.3. Define the following relation between Cauchy sequences:

$$x_n \sim y_n \qquad \text{if and only if} \qquad x_n - y_n \to 0$$

Check that this is an equivalence relation, meaning

- $x_n \sim x_n$
- $x_n \sim y_n \Rightarrow y_n \sim x_n$
- $x_n \sim y_n$ and $y_n \sim z_n \Rightarrow x_n \sim z_n$

2. The set of real numbers

2.1. Notation. Let $\mathbb{R}$ the set of equivalence classes of Cauchy sequences of rational numbers. For $x \in \mathbb{R}$ and x_n a Cauchy sequence, we write $\lim_n x_n = x$ if x_n is in the equivalence class of x.

2.2. Function $j : \mathbb{Q} \to \mathbb{R}$ define by $j(r) = (r, r, \dots)$. Check that this is well defined and injective.