CAUCHY SEQUENCES OF RATIONAL NUMBERS

1. Definitions

1.1. Definition. A sequence \( \{x_n\}_{n \geq 1} \) of rational numbers is said to be Cauchy if it satisfies the following property:

\[
\forall \epsilon > 0, \exists N > 0 \text{ such that } m, n \geq N \Rightarrow |x_m - x_n| \leq \epsilon
\]

1.2. Definition. We say that \( x_n \to 0 \) if it has the following property:

\[
\forall \epsilon > 0, \exists N > 0 \text{ such that } n \geq N \Rightarrow |x_n| \leq \epsilon
\]

1.3. Define the following relation between Cauchy sequences:

\( x_n \sim y_n \) if and only if \( x_n - y_n \to 0 \)

Check that this is an equivalence relation, meaning

- \( x_n \sim x_n \)
- \( x_n \sim y_n \Rightarrow y_n \sim x_n \)
- \( x_n \sim y_n \) and \( y_n \sim z_n \Rightarrow x_n \sim z_n \)

2. The set of real numbers

2.1. Notation. Let \( \mathbb{R} \) the set of equivalence classes of Cauchy sequences of rational numbers. For \( x \in \mathbb{R} \) and \( x_n \) a Cauchy sequence, we write \( \lim_{n} x_n = x \) if \( x_n \) is in the equivalence class of \( x \).

2.2. Function \( j : \mathbb{Q} \to \mathbb{R} \) define by \( j(r) = (r, r, \ldots) \). Check that this is well defined and injective.