COMPACT SETS, CONNECTED SETS AND CONTINUOUS FUNCTIONS

1. Definitions

1.1. $D \subset \mathbb{R}$ is **compact** if and only if for any given open covering of D we can subtract a finite sucovering. That is, given $(G_\alpha)\alpha \in \mathcal{A}$ a collection of open subsets of $\mathbb{R}$ ($\mathcal{A}$ an arbitrary set of indices) such that $D \subset \cup_{\alpha\in\mathcal{A}} G_\alpha$, then there exists finitely many indices $\alpha_1, \ldots, \alpha_N \in \mathcal{A}$ such that $D \subset \cup_{i=1}^{N} G_{\alpha_1}$.

1.2. Let D an arbitrary subset of $\mathbb{R}$. Then $A \subset D$ is **open in** D (or relative to D, or D-open) if and only if there exists G open subset of $\mathbb{R}$ such that $D = G \cap D$. Similarly we can define the notion of D-closed sets. Note that D is both open and closed in D, and so is $\emptyset$.

1.3. $D \subset \mathbb{R}$ is **connected** if and only if $\emptyset$ and D are the only subsets of D which are both open in D and closed in D. In other words, if $D = A \cup B$ and A, B are disjoint D-open subsets of D, then either $A = \emptyset$ or $B = \emptyset$.

1.4. Let $D \subseteq \mathbb{R}$, $a \in D$ a fixed element and $f : D \to \mathbb{R}$ an arbitrary function. By definition, f is continuous at a if and only if the following property holds

$$\forall\, \epsilon > 0, \quad \exists\, \delta_a(\epsilon) > 0 \text{ such that } |x - a| < \delta_a(\epsilon) \wedge x \in D \Rightarrow |f(x) - f(a)| < \epsilon$$

The last implication can be re-written in terms of sets as follows:

$$f\Big(B_a\big(\delta_a(\epsilon)\big) \cap D\Big) \subseteq B_{f(a)}(\epsilon)$$

Here we use the notation $B_x(r) := (x - r, x + r)$.

1.5. Sequence characterization of continuity: f is continuous at a iff for any sequence $(x_n)_{n\geq 1}$ such that $x_n \in D, n \geq 1$ and $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x_n) = f(a)$.

1.6. We say that $f : D \to \mathbb{R}$ is continuous on D (or simply say continuous) if and only if f is continuous at every $a \in D$.

2. Characterization of continuous functions using preimages

2.1. **Theorem.** Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ a function. Then the following propositions are equivalent:
a) f is continuous (on D).
b) $\forall\, G \subseteq \mathbb{R}$ open, $f^{-1}(G)$ is open in D.
c) $\forall\, F \subseteq \mathbb{R}$ closed, $f^{-1}(F)$ is closed in D.

Proof. $a \Rightarrow b$. Let $G \subseteq \mathbb{R}$ open. Pick $a \in f^{-1}(G)$. Then $f(a) \in G$, and since G is open, there must exist $\epsilon > 0$ such that $B_{f(a)}(\epsilon) \subseteq G$. By continuity, corresponding to this $\epsilon > 0$ there exists $\delta > 0$ such that $f\Big(B_a(\delta) \cap D\Big) \subset B_{f(a)}(\epsilon)$. But this places the entire set $B_a(\delta) \cap D$ inside $f^{-1}(G)$:

$$B_a(\delta) \cap D \subseteq f^{-1}(G)$$

Writing now $\delta = \delta_a$ to mark the dependence of δ on a, and varying $a \in f^{-1}(G)$, we obtain

$$f^{-1}(G) = \big(\cup_{a\in f^{-1}(G)} B_a(\delta_a)\big) \cap D$$

which shows that $f^{-1}(G)$ is open in D.
$b \Leftrightarrow c$. Let $F \subseteq \mathbb{R}$ a closed set, which is equivalent to saying that $G = \mathcal{C}F$ (the complement in $\mathbb{R}$) is open. Then

$$f^{-1}(F) = \{x \in D | f(x) \in F\} = \{x \in D | f(x) \notin G\} = D - f^{-1}(G)$$

Since the complement of a D-open subset of D is D-closed, it means that $f^{-1}(F)$ is closed in D if and only if $f^{-1}(G)$ is open in D.
$c \Rightarrow a$: exercise.

2.2. Using this characterization, we can prove for example that the composition of continuous functions is a continuous function.

Proposition. Assume $f : D \to \mathbb{R}$ is continuous, $g : E \to \mathbb{R}$ is continuous, and $f(D) \subseteq E$. Then the function $h := g \circ f : D \to \mathbb{R}$ defined by $h(x) = g(f(x))$ is continuous.

Proof. Let $G \subseteq \mathbb{R}$ an open set. Then $h^{-1}(G) = f^{-1}\big(g^{-1}(G)\big)$. But $g^{-1}(G) = V \cap E$, for some open set $V \subseteq \mathbb{R}$. But then $h^{-1}(G) = f^{-1}(V \cap E) = f^{-1}(V)$ is open in D, so h is continuous.

2.2.1. Example. Assume $f : D \to \mathbb{R}$ is a continuous function, such that $f(x) \neq 0$, $\forall x \in D$. Then $h : D \to \mathbb{R}$ given by $h(x) = 1/f(x)$, is continuous as well. Proof: $g : \mathbb{R} - \{0\} \to \mathbb{R}$, $g(x) = 1/x$ is continuous (proved in class), $f(D) \subseteq \mathbb{R} - \{0\}$, hence $h = g \circ f$ is continuous.

3. General properties continuous functions

3.1. **Theorem.** A continuous function maps compact sets into compact sets.

Proof. In other words, assume $f : D \to \mathbb{R}$ is continuous and D is compact. Then we need to prove that the image $f(D)$ is a compact subset of $\mathbb{R}$. For that, we consider an arbitrary open covering $f(D) \subseteq \cup_\alpha G_\alpha$ of $f(D)$ and we will try to find a finite subcovering. Taking the preimage we have $D \subseteq \cup_\alpha f^{-1}(G_\alpha)$. But $f^{-1}(G_\alpha)$ is open in D, so there must exist $V_\alpha \subseteq \mathbb{R}$ open such that $f^{-1}(G_\alpha) = V_\alpha \cap D$. Then $D \subseteq \cup_\alpha (V_\alpha \cap D)$ which simply means that $D \subseteq \cup_\alpha V_\alpha$. We thus arrived at an open covering of D, so there must exist finitely many indices $\alpha_1, \ldots, \alpha_N$ such that $D \subseteq \cup_{i=1}^N V_{\alpha_i}$, which implies the equality $D = \cup_{i=1}^N (V_{\alpha_i} \cap D) = \cup_{i=1}^N f^{-1}(G_{\alpha_i})$. But this implies in turn that $f(D) \subseteq \cup_{i=1}^N G_{\alpha_i}$. So $f(D)$ is compact.

3.2. **Theorem.** A continuous function maps connected sets into connected sets.

In other words, assume $f : D \to \mathbb{R}$ is continuous and D is connected. Then $f(D)$ is connected as well.

Proof. Assume $f(D)$ is not connected. Then there must exist A, B disjoint, non-empty, subsets of $f(D)$, both open relative to $f(D)$, such that $f(D) = A \cup B$. Being open relative to $f(D)$ simply means there exists $U, V \subseteq \mathbb{R}$ open such that $A = f(D) \cap U$, $B = f(D) \cap V$. So $f(D) \subseteq U \cup V$. But this implies that $D \subset f^{-1}(U) \cap f^{-1}(V)$. Since U, V are open, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are open relative to D. But they are also disjoint (why?). Since D is connected, it follows that at least one of them, say $f^{-1}(U)$, is empty. But $A \subseteq U$, so this forces $f^{-1}(A) = \emptyset$ as well, which is impossible unless $A = \emptyset$ (note that A is a subset of the image of f), contradiction.

3.3. **Theorem.** A continuous function on a compact set is uniformly continuous.

Proof. Assume D compact and $f : D \to \mathbb{R}$ continuous. Given $\epsilon > 0$ we need to find $\delta(\epsilon) > 0$ such that if $x, y \in D$ and $|x - y| < \delta(\epsilon)$, then $|f(x) - f(y)| < \epsilon$.

From the definition of continuity, given such $\epsilon > 0$ and $x \in D$, there exists $\delta_x(\epsilon)$ such that if $|y - x| < \delta_x(\epsilon)$, then $|f(y) - f(x)| < \epsilon$. Clearly $D \subseteq \cup_{x \in D} B_x(\frac{1}{2}\,\delta(\epsilon/2))$. From this open covering we can extract a finite subcovering (D is compact!), meaning there must exists finitely many $x_1, x_2, \ldots, x_N \in D$ such that $D \subseteq \cup_{i=1}^N B_{x_i}(\frac{1}{2}\,\delta_{x_i}(\epsilon/2))$.

Let now $\delta(\epsilon) = \min\{\frac{1}{2}\,\delta_{x_1}(\epsilon/2), \ldots, \frac{1}{2}\,\delta_{x_N}(\epsilon/2)\}$. We will show that $\delta(\epsilon)$ does the job.

Take $y, z \in D$ arbitrary such that $|y - z| < \delta(\epsilon)$. The idea is that y will be near some x_j, which in turn places z near that same x_j. But that forces both $f(y)$, $f(z)$ to be close to $f(x_j)$ (by continuity at x_j), and hence close to each other.

Since $y \in D$, there must exist some j, $1 \leq j \leq N$ such that $y \in B_{x_j}(\frac{1}{2}\delta_{x_j}(\epsilon/2))$. Thus

- $|y - x_j| < \frac{1}{2}\,\delta_{x_j}(\epsilon/2)$
- but $|y - z| < \delta(\epsilon) \leq \frac{1}{2}\,\delta_{x_j}(\epsilon/2)$

By the triangle inequality it follows that $|z - x| < \delta_{x_j}(\epsilon/2)$. So y, z are within $\delta_{x_j}(\epsilon/2)$ of x. This implies that

- $|f(y) - f(x_j)| < \epsilon/2$
- $|f(z) - f(x_j)| < \epsilon/2$

By the triangle inequality once again we have $|f(y) - f(z)| < \epsilon$.

Alternative proof, using sequences. Assume f is not uniformly continuous, meaning that there exists $\epsilon > 0$ such that no $\delta > 0$ does the job. Checking what this means for $\delta = \frac{1}{n}$, we see that for any such $n \geq 1$ there exist $x_n, y_n \in D$ such that $|x_n - y_n| < \frac{1}{n}$ and yet $|f(x_n) - f(y_n)| > \epsilon$. However D is compact, in particular any sequence in D has a convergent subsequence whose limit belongs to D. Applying this principle twice we find that there must exist $n_1 < n_2 < \ldots$ such that the subsequences $(x_{n_k})_{k \geq 1}$ and $(y_{n_k})_{k \geq 1}$ are convergent, and $x = \lim_{k \to \infty} x_{n_k} \in D$, $y = \lim_{k \to \infty} y_{n_k} \in D$. We have the following:

- By construction, $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$. Taking the limit, we find $x = y$.
- By continuity, $\lim_{k\to\infty} f(x_{n_k}) = f(x)$, since $x \in D$. Also $\lim_{k\to\infty} f(y_{n_k}) = f(y)$.
- Also by construction, $|f(x_{n_k}) - f(y_{n_k})| > \epsilon$ hence in the limit, $|f(x) - f(y)| \geq \epsilon$.

We thus reach a contradiction.

4. Characterization of the compact sets and the connected sets of $\mathbb{R}$

4.1. **Theorem.** In $\mathbb{R}$, compact = bounded & closed.

We prove more, namely:

4.2. **Proposition.** Let $D \subseteq \mathbb{R}$. Then the following propositions are equivalent:
a) D is compact
b) D is bounded and closed
c) Every sequence in D has a convergent subsequence whose limit belongs to D.

Proof. $a \Rightarrow b$. $D \subseteq \mathbb{R} = \cup_{n=1}\infty(-n, n)$ is an open covering of D. Hence $\exists N \geq 1$ such that $D \subseteq \cup_{n=1}^{N}(-n, n) = (-N, N)$. This shows D is bounded. To prove D is closed, we prove that $\mathbb{R} - D$ is open. Let $y \in \mathbb{R} - D$. Then $D \subseteq \cup_{n=1}^{\infty}(\mathbb{R} - [y - \frac{1}{n}, \frac{1}{n}])$ (why?). This open covering must have a finite subcovering, so $\exists N \geq 1$ such that $D \subseteq \mathbb{R} - [y - \frac{1}{N}, y + \frac{1}{N}]$. But this implies that $(y - \frac{1}{N}, y + \frac{1}{N}) \subseteq \mathbb{R} - D$. But y was chosen arbitrary in $\mathbb{R} - D$, so this set is open, and hence D itself is closed.
$b \Rightarrow c$. This has to do with the fact that every bounded sequence has a convergent subsequence.
$c \Rightarrow b$. Here one shows $D = \overline{D}$ and this has to do with the fact that $\overline{D}$ is the set of limits of convergent sequences of D, etc...
$c \Rightarrow a$ Let $D \subseteq \cup_{k=1}^{\infty} G_k$ be an arbitrary open covering of D. (Note: a covering by a countable collection of open sets is not the most general infinite open covering one can imagine, of course; we need an intermediate step to prove that from any open covering of D we can extract a countable subcovering, and this has to do with the fact that $\mathbb{R}$ admits a countable dense set. Read the details of this step in the textbook.) Let's prove there exists $n \geq 1$ such that $D \subseteq \cup_{n=1}^{\infty} G_k$. Assume this was not the case. Then $\forall n \geq 1$, there exists $x_n \in D - \cup_{k=1}^{n} G_k$. But x_n is a sequence in D, so it must have a convergent subsequence, call it $(x_{n_j})_{j\geq 1}$, with limit in D, so $\lim_{j\to\infty} x_{n_j} = a \in D$. But a belongs to one of the $G_i's$, say $a \in G_N$. Since G_N is open, it follows that $x_{n_j} \in G_N$, for $j \geq j_0$ (j large enough). In particular this shows that for j large enough (larger than j_0 and larger than N) we have $x_{n_j} \in G_N \subseteq \cup_{k=1}^{n_j}$, since $n_j \geq j > N$. This contradicts the defining property of x_n's.

4.3. **Theorem.** $\mathbb{R}$ is connected.

Proof. This is really the statement that $\emptyset$ and $\mathbb{R}$ itself are the only subsets of $\mathbb{R}$ which are both open and closed. To prove this, let E be a non-empty subset of $\mathbb{R}$ with this property. We'll prove that $E = \mathbb{R}$. For that, take an arbitrary $c \in \mathbb{R}$. To prove that $c \in E$, we assume that $c \notin E$ and look for a contradiction. Since E is non-empty, it follows that E either has points to the left of c or to the right of c. Assume the former holds.
α) Consider the set $S = \{x \in E | x < c\}$. By construction, S is bounded from above (c is an upper bound for S). Therefore we can consider $y = l.u.b.(S) \in \mathbb{R}$.
β) Input: E is closed. Then $S = E \cap (-\infty, c]$ is also closed. Then $y \in \bar{S} = S$, so $y < c$.
γ) Input: E is open. $y \in S \subseteq E$ and E is open, this means that there exists $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subseteq E$. Choose ϵ small enough so that $\epsilon < c - y$. In that case $z = y + \frac{\epsilon}{2} \in (y - \epsilon, y + \epsilon) \subseteq E$ is an element of E which the properties

- $z < c$, hence $z \in S$
- $z > y$

which is in contradiction with the defining property of y.

4.4. **Theorem.** The only connected subsets of $\mathbb{R}$ are the intervals (bounded or unbounded, open or closed or neither).

Proof. First we prove that a connected subset of $\mathbb{R}$ must be an interval.
Step 1. Let $E \subseteq \mathbb{R}$ a connected subset. We prove that if $a \in b \in E$, then $[a, b] \subseteq E$. In other words, together with any two elements, E contains the entire interval between them. To see this, let c a real number between a and b. Assume $c \notin E$. Then $E = A \cup B$, where $A = (-\infty, c) \cap E$ and $B = (c, +\infty) \cap E$. Note that A and B are disjoint subsets of D, both open relative to D. Since E is connected, at least one of them should be empty, contradiction, since $a \in A$ and $b \in B$. Thus $c \in E$.
Step 2. To show that E is actually an interval, consider $\inf E$ and $\sup E$. Case one. E is bounded. Then

$m = \inf E, M = \sup E \in \mathbb{R}$, and clearly $E \subseteq [m, M]$. On the other hand, for any given $x \in (m, M)$, there exists $a, b \in E$ such that $a < x < b$. That's because $m, M \in \overline{E}$ and one can find elements of E as close to m (resp. M) as desired (draw a picture with the interval (m, M) and place a point x inside it). But then $[a, b] \subseteq E$, and in particular $x \in E$. Since x was chosen arbitrarily in (m, M), we must have $(m, M) \subseteq E \subseteq [m, M]$, so E is definitely an interval. Case two: E is unbounded. With a similar argument, show that E is an unbounded interval.
Conversely, we need to show that intervals are indeed connected sets. The proof is almost identical to that in the case where the interval is $\mathbb{R}$ itself.

5. Corollaries: theorems for continuous functions on $\mathbb{R}$

5.1. **Theorem.** Let $D \subseteq \mathbb{R}$ compact and $f : D \to \mathbb{R}$ a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \le f(x) \le f(y_2)$, $\forall x \in D$.

Proof. $f(D)$ is a compact subset of $\mathbb{R}$, so it is bounded and closed. This implies that $g.l.b(f(D)) \in f(D)$ and $l.u.b.(f(D)) \in f(D)$ as well. But then there must exist $y_1, y_2 \in D$ such that $f(y_1) = g.l.b.f(D)$ and $f(y_2) = l.u.b.f(D)$. But this implies $f(D) \subseteq [f(y_1), f(y_2)]$ and we are done.

Note: one uses the notation $\sup_{x\in D} f(x)$ to denote the l.u.b. of the image of D. In other words, $\sup_{x\in D} f(x) = l.u.b.\{f(y) \mid y \in D\}$. The theorem says that if D is compact and f is continuous, then $\sup_{x\in D} f(x)$ is finite, and more over that there exists $y_1 \in D$ such that $f(y_1) = \sup_{x\in D} f(y)$. If the domain is not compact, one can find examples of continuous functions such that either i) $\sup f = +\infty$ or such that ii) $\sup f$ is a real number but not in the image of f.
For case i), take $f(x) = 1/x$ defined on $(0, 1]$. For case ii), take $f(x) = x$ defined on $[0, 1)$.

5.2. **Theorem.** A continuous (real-valued) function defined on an interval in $\mathbb{R}$ has the intermediate value property.

Proof. Assume E is an interval in $\mathbb{R}$ and $f : E \to \mathbb{R}$ a continuous function. Let $a, b \in E$ (say $a < b$) and y a number between $f(a)$ and $f(b)$. The intermediate value property is the statement that there exists c between a and b such that $f(c) = y$. But this follows immediately from the fact that $f(E)$ is an interval. (E is an interval in $\mathbb{R} \Rightarrow E$ is connected $\Rightarrow f(E)$ is a connected subset of $\mathbb{R} \Rightarrow f(E)$ is an interval in $\mathbb{R}$).

5.3. **Problem.** Prove that there does not exist a continuous, bijective function $f : [0, 1) \to \mathbb{R}$.

Answer. Assume such a function exists. Then $f([0, \frac{1}{2}])$ is a compact subset of $\mathbb{R}$, so it is bounded. There must exist $N > 0$ such that $f([0, \frac{1}{2}]) \subseteq [-N, N]$. But f is assumed to be surjective, so there must exist $a, b \in [0, 1)$ such that $f(a) = -N - 1$ and $f(b) = N + 1$. Certainly $a, b \in (\frac{1}{2}, 1)$. By the intermediate value property $[-N - 1, N + 1] \subseteq f((\frac{1}{2}, 1))$. In particular $f(0) \in f((\frac{1}{2}, 1))$, which means that there exists $c \in (\frac{1}{2}, 1)$ such that $f(0) = f(c)$. But this means f is not injective, contradiction.