

## 1. SEQUENCES OF FUNCTIONS

1.1. **Theorem.** Assume  $f_n$  is a sequence of continuous functions all of them defined on a same open interval  $I$ . Assume  $f_n \rightarrow f$  uniformly on  $I$ . Then:

a)  $f$  is continuous on  $I$ .

b)  $\lim_n \int_a^b f_n(t)dt = \int_a^b f(t)dt$  for any compact interval  $[a, b] \subset I$ .

Note. Fix  $a \in I$  and let  $F_n(x) = \int_a^x f_n(t)dt$ ,  $F(x) = \int_a^x f(t)dt$ , for  $x \in I$ . Part b) of the theorem says that  $F_n \rightarrow F$  on  $I$  (pointwise). Question: Is the convergence uniform?

1.2. **Theorem.** Assume  $f_n$  is a sequence of differentiable functions on  $I$  such that:

- $f_n \rightarrow f$  converges pointwise on  $I$
- $f'_n$  converges uniformly on  $I$

Then  $f(x) := \lim_n f_n(x)$  is differentiable and  $f'(x) = \lim_n f'_n(x)$ ,  $\forall x \in I$ .

Exercise: prove this theorem.

## 2. SERIES OF FUNCTIONS

2.1. Assume  $I$  is an (open) interval and  $u_n(x)$  is a sequence of functions defined on  $I$ .

Definition: we say that the series  $\sum_{n=1}^{\infty} u_n(x)$  converges (simply, uniformly) to the function  $F : I \rightarrow \mathbb{R}$  if and only if the sequence of functions (partial sums)  $S_n(x)$  converges (pointwise, uniformly) to  $F$ , where:

$$n \geq 1, S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x), \quad \forall x \in I$$

2.2. **Theorem.** Assume that there exist a sequence of positive real numbers  $a_n$  such that:

- $|u_n(x)| \leq a_n, \quad \forall n \geq 1, \forall x \in I$
- $\sum_{n=1}^{\infty} a_n < +\infty$

Then: the series  $\sum_{n=1}^{\infty} u_n(x)$  converges uniformly on  $I$ .