

PRACTICE PROBLEMS: FUNCTIONS

1. State the intermediate value theorem for real-valued continuous function. Give an explanation (proof) based on the notion of connected sets.
2. Prove that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has the intermediate value property if and only if it maps intervals onto intervals. Is such a function necessarily continuous?
3. Consider $g : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.
 - a) Prove that g is not continuous.
 - b) Prove that g has the intermediate value property, i.e. g maps intervals into intervals.
4. Prove that a continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$ which takes only rational values is constant.
5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ a continuous, injective function. Prove that f is monotone.
6. Let $f : [0, 1] \rightarrow [0, 1]$ a continuous function. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$.
8. Assume $h : [0, 1] \rightarrow (0, \infty)$ is a continuous function. Prove that there exists $\delta > 0$ such that $f(x) > \delta$, $\forall x \in [0, 1]$.
9. Let D an open subset of \mathbb{R} and $f : D \rightarrow \mathbb{R}$ a continuous function which doesn't vanish anywhere. Prove that the function $g : D \rightarrow \mathbb{R}$, $g(x) = \frac{1}{f(x)}$ is continuous as well.
10. Let $f : D \rightarrow \mathbb{R}$ a continuous function, where the domain D is a compact subset of \mathbb{R} . Prove that $f(D)$ is compact.
11. True/false: there exists a continuous, bijective function $f : [0, 1] \rightarrow (2, 3)$.
12. True/false: there exists a continuous, bijective function $g : (2, 3) \rightarrow [0, 1]$.
13. Give an example of a bounded function $f : [0, 1] \rightarrow \mathbb{R}$ which does not attain its maximum on $[0, 1]$. Can such a function be continuous? Explain your answer.
15. Assume $f : [0, \infty) \rightarrow [0, +\infty)$ a positive is continuous and $\lim_{x \rightarrow +\infty} f(x) = 0$. Prove that there exists $x_0 \geq 0$ such that $f(x) \leq f(x_0)$, $\forall x \geq 0$.
16. Assume the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $x = 0$ and x_n is a convergent sequence with $\lim x_n = 0$. Prove (using the definitions) that $\lim f(x_n) = f(0)$.
17. True/false: if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$, then $f(\overline{A}) \subseteq \overline{f(A)}$.
18. True or false: If $f(x) = o(x^2)$ as $x \rightarrow 0$, then $f(x) = O(x^3)$ as $x \rightarrow 0$.
19. Assume $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\lim_{x \rightarrow +\infty} g(x) = +\infty$. Prove the existence of a monotone increasing sequence x_n such that $\lim_{n \rightarrow \infty} (g(x_{n+1}) - g(x_n)) = +\infty$.
22. Assume I is a bounded interval and $f : I \rightarrow \mathbb{R}$ is continuous such that image $f(D)$ is an open interval. Is D necessarily an open interval?
26. Let $f : [1, +\infty) \rightarrow \mathbb{R}$, $f(x) = x \sin x$. Determine the image of f
27. Prove that there exists $b \geq 2006$ such that $b \sin b = 2005$.
23. Assume that $f(x)$ is monotone. Prove that $\lim_{x \rightarrow +\infty} f(x)$ exists. Moreover, if f is bounded the limit is finite.
24. Assume the function $f(x)$ is continuous at $x = 0$ and $f(0) = 1$. Prove (using the ϵ, δ -definitions) that $f(x) \geq \frac{1}{2}$ in a neighborhood of $x = 0$.
25. Assume $f : [0, 2] \rightarrow \mathbb{R}$ is a monotone bounded function. Prove that the limit $\lim_{x \rightarrow 1, x < 1} f(x)$ exists.
28. True or false: if the pre-image of every closed interval through a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a closed interval, then f is continuous.
29. Assume f is C^1 and $f'(0) = 1$. Prove that f is monotone in a neighborhood of $x = 0$.
30. a) $h : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$, $h(x) = x \cos x - \sin x$. Find $\sup_{x \in [0, \frac{\pi}{2}]} h(x)$.
b) Prove that $1 \leq \frac{\sin x}{x} \leq \frac{2}{\pi}$, $\forall x \in (0, \frac{\pi}{2}]$.

31. Let $n \geq 1$ and $f : (0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = x \ln(\frac{ne}{x})$.

a) Determine $\sup_{x>0} f(x)$.

b) Prove the inequality $(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}$.

32. Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that f' vanishes at the irrational numbers. Prove that f is constant.

33. Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary function, and $x_0 \in \mathbb{R}$. Prove that f is continuous at x_0 iff $\text{Osc}(f, x_0) = 0$.

34. Let $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q}, \text{ gcd}(p, q) = 1, q > 0 \\ 0, & x \notin \mathbb{Q} \end{cases}$.

Determine $C(f)$, the set where f is continuous.

35. For $E \subseteq [0, 1]$ an arbitrary subset of $[0, 1]$, the indicator function is defined by $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$.

Determine $C(\chi_E)$, the set where χ_E is continuous.

36. Assume $f : [0, 1] \rightarrow \mathbb{R}$ is a bounded function and $\delta > 0$ some positive real number. Prove that $\{x \in [a, b] : \text{Osc}(f, x) < \delta\}$ is open.

37. Assume $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function. Prove that $\mathcal{D}(f)$ is an F_σ -set.

38. Assume $f : [0, 1] \rightarrow \mathbb{R}$ is a bounded function that has a jump discontinuity at $a \in (0, 1)$. I.e., the limits $f(a+) := \lim_{x \rightarrow a, x > a} f(x)$ and $f(a-) := \lim_{x \rightarrow a, x < a} f(x)$ exist and $f(a-) \neq f(a+)$. Determine $\text{Osc}(f, a)$.