PRACTICE PROBLEMS: INTEGRATION

1. Assume f is RI on $[a,b]$. Prove using the definitions that there exist $\delta(\epsilon) > 0$ such that $\mathrm{Osc}(f,P) \leq \epsilon$ whenever $\epsilon > 0$.

2. Assume $f : [a,b] \to \mathbb{R}$ is RI. Prove using the definitions that $S^+(f) = S^-(f)$.

3. Assume $f : [a,b] \to \mathbb{R}$ is a bounded function and $\delta > 0$. Prove that the set $A(\delta) = \{x : \mathrm{Osc}(f,x) \geq \delta$ is closed.

4. Assume $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function. Let P an arbitrary partition of the (bounded) interval $[a,b]$. Prove that there exists a constant C (depending on f) such that $\mathrm{Osc}(f,P) \leq C\|P\|$ for an arbitrary partition P.

5. Assume A is a finite set. Prove that A has content zero.

6. Assume A is a countable set. Prove that A is negligible.

7. True/false: a Riemann integrable function $f : [0,1] \to \mathbb{R}$ is bounded.

8. True/false: a Riemann integrable function $f : [0,1] \to \mathbb{R}$ achieves its maximum on $[0,1]$.

9. Assume $f : [0,1] \to \mathbb{R}$ is bounded and continuous except at finitely many points. Prove that f is Riemann integrable.

10. Assume $f : [0,1] \to \mathbb{R}$ is Riemann integrable and $g : [0,1] \to \mathbb{R}$ is a real function such that $f(x) = g(x)$ for all $x \in [0,1] - E$, where $E \subseteq [0,1]$ is a finite set. Prove that g is Riemann integrable and $\int f = \int g$. points in $[0,1]$.

11. Assume $f : [0,1] \to \mathbb{R}$ is bounded and monotone increasing. Prove that f is Riemann integrable.

12. True or false: if $f : [0,1] \to \mathbb{R}$ is Riemann integrable and $F(x) = \int_0^x f(t)dt$, then F is continuous at $x = \frac{1}{2}$.

13. Assume $f : [0,2] \to \mathbb{R}$ is monotone and continuous everywhere except at the point $x = 1$. Let $F(x) = \int_0^x f(t)dt$. True or false: F is continuous at $x = 1$.

14. True or false: if $f : [0,1] \to \mathbb{R}$ is Riemann integrable and $F(x) := \int_0^x f(t)dt$, then F is differentiable at infinitely many points.

16. Assume $f : [0,1] \to \mathbb{R}$ is a bounded function. Prove that $\mathcal{D}(f)$ is an F_σ-set.

17. Prove that $\mathbb{R} - \mathbb{Q}$ is a G_δ-set.

18*. Argue that it is impossible to construct a function which is continuous exactly at the rational numbers.

19. Let Q a partition of $[0,1]$ whose division points are: $x_0 = 0, x_1 = 1/4, x_2 = 1/3, x_3 = 1/2, x_4 = 1$. Assume moreover that $f : [0,1] \to \mathbb{R}$ is a bounded function. Determine a positive integer N such that $\mathrm{Osc}(f,P_n) \leq 3\,\mathrm{Osc}(f,Q)$, for all $n \geq N$. Here P_n denotes the standard partition of $[0,1]$ of norm $1/n$.

21. Let $E \subseteq [0,1]$. Give a sufficient condition for E to satisfy so that the indicator function $\chi_E : [0,1] \to \mathbb{R}$ is Riemann integrable.

22. Assume that f is a positive, monotone function such that the improper integral $\int_0^\infty f(x)dx$ converges. Prove that $\lim_{x\to+\infty} f(x) = 0$.

23. For $n \geq 1$, let $y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$. Prove that y_n is convergent and compute $\lim y_n$.

24. For $n \geq 1$, let $z_n = \frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}$. Prove that z_n is convergent and compute $\lim z_n$.

25. Compute $\displaystyle\lim_{n\to+\infty} \frac{1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{3/2}}$.