1. POWER SERIES.

1.1. Definition: a series of functions of the form $\sum_{n=1}^{\infty} a_n x^n$ where $a_1, a_2, \dots,$ are given (complex) numbers, is called a power series (in x).
Partial sums: $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial of degree (at most) n.

Example. For $\sum_{n=1}^{\infty} n x^{n^2}$, the coefficient a_n is given by $a_n = \begin{cases} 0, & n \text{ is not a square} \\ \sqrt{n}, & n \text{ is a square} \end{cases}$. In this example $S_5(x) = x + 2x^4$.

1.2. Definition: the set of real (or complex) values of x for which the power series converges is called the domain of convergence. We'll denote it by $\mathcal{D}$: $\mathcal{D} = \{x \in \mathbb{R} : \sum_n a_n x^n \text{ convergent}\}$.
Note: $0 \in \mathcal{D}$, so $\mathcal{D}$ is never empty.

2. RADIUS OF CONVERGENCE

2.1. Definition: the radius of convergence R is the positive number (or $+\infty$) given by

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}$$

2.2. **Theorem** [Cauchy-Hadamard]. The power series $\sum_{n=1}^{\infty} a_n x^n$:
a) converges absolutely for $|x| < R$.
b) diverges for $|x| > R$
c) Moreover, the convergence is uniform on $|x| \le R - \delta$, for any $\delta > 0$.

Note: the case of $|x| = R$ is not covered by the theorem, and needs to be analyzed separately.
Note: if $R = +\infty$ then $\mathcal{D} = \mathbb{R}$.

3. HOW TO COMPUTE R.

In most of the specific cases $|a_n|^{1/n}$ is actually convergent, in which case $\limsup = \lim$.

3.1. First method: keep in mind the following facts:

- (Average) $x_n \to L \Rightarrow \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \to L$
- (Multiplicative analogue) $x_n > 0$, $x_n \to L \Rightarrow \lim_n (x_1 x_2 \dots x_n)^{1/n} = L$.
- Corollary: if $x_n > 0$ and $x_{n+1}/x_n \to L$, then $x_n^{1/n} \to L$.

3.2. Second method: let $b_n = \log(|a_n|^{1/n}) = \frac{1}{n} \log |a_n|$. Check if b_n has a limit, then $|a_n|^{1/n} \to e^{\lim b_n}$.

3.3. Example. $\sum_{n=1}^{\infty} (-1)^n n 2^n x^n$.
Coefficients: $a_n = (-1)^n n 2^n$; $\quad |a_n|^{1/n} = 2n^{1/n}$. What is $\limsup_{n \to +\infty} 2n^{1/n}$?
Take log: $b_n = \log |a_n|^{1/n} = \ln 2 + \frac{\ln n}{n} \to \ln 2$. Then $|a_n|^{1/n} \to e^{\log 2} = 2$.
Radius: $1/R = \lim |a_n|^{1/n} = 2 \Rightarrow R = 1/2$.
Hence $(-1/2, 1/2) \subseteq \mathcal{D}$.
Check "endpoints" separately:
At $x = -1/2$: $\sum a_n x^n = \sum n$ divergent.
At $x = 1/2$: $\sum a_n x^n = \sum (-1)^n n$ divergent.
Therefore $\mathcal{D} = (-1/2, 1/2)$.

3.4. Example. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.
Coefficients: $a_n = 1/n!$, $\quad |a_n|^{1/n} = 1/(n!)^{1/n}$.
Take log: $b_n = \log |a_n|^{1/n} = -\frac{\log n!}{n}$. Not so clear what $\lim b_n$ is.
Alternative: $|a_{n+1}|/|a_n| = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0$, hence $|a_n|^{1/n} \to 0$ as well.
Conclusion: $R = +\infty$, $\mathcal{D} = \mathbb{R}$.

3.5. Example. $\sum n x^{n^2}$. Here $|a_n|^{1/n} = \begin{cases} n^{1/2n}, & n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$. In this case $|a_n|^{1/n}$ is *not* convergent, but $\limsup |a_n|^{1/n} = 1$. (Justify this step.) So $R = 1$. It is not hard to see that $\mathcal{D} = (-1, 1)$.

4. HOW TO USE POWER SERIES

4.1. Assume R is the radius of convergence of the power series $\sum a_n x^n$. Since $n^{1/n} \to 1$, it is not hard to see that the two series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ have the same radius of convergence R. Since the power series converges uniformly on compact intervals of the form $[-r, r] \subset (-R, R)$, we have

Theorem. Let $f(x) = \sum a_n x^n$ for $x \in \mathcal{D}$. Then:
a) $f(x)$ is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $\forall x \in (-R, R)$.
b) $\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$, $\forall x \in (-R, R)$.

Note: applying this theorem repeatedly we see that the sum of a power series is a smooth (infinitely many times differentiable) function inside the radius of convergence, $f \in C^{\infty}\big((-R, R)\big)$.

4.2. Example. Let $f(x) = \sum \frac{x^n}{n!}$, for $x \in \mathbb{R}$. Then f is C^1 on $\mathbb{R}$, and $f'(x) = \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1}$, that is $f'(x) = f(x)$. Moreover $f(0) = 1$. The only function that satisfies this equation is $f(x) = e^x$. In particular, $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.

4.3. Example. Let $f(x) = \sum (-1)^n x^{2n}$, for $|x| < 1$ ($R = 1$). Then clearly $f(x) = \frac{1}{1+x^2}$.
Integrate: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \int_0^x \frac{1}{1+t^2} dt = \tan^{-1}(x)$, for $|x| < 1$.

4.4. Example. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$, for $|x| < 1$.

5. ABEL'S THEOREM

5.1. Note that if one could take $x = 1$ in the previous two examples, we would obtain nice identities:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

This step is justified by the following theorem:

Theorem[Abel]. Let R the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, and let $f(x)$ the sum of the power series inside the radius of convergence. If $R \in \mathcal{D}$, then $\lim_{x \to R-} f(x) = \sum_{n=0}^{\infty} a_n R^n$. (The limit $f(R-)$ exists and equals the sum of the series at $x = R$.)

Note: the theorem is works just the same for $-R$ instead of R.