

RIEMANN INTEGRATION

1. SET-UP

$a, b \in \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function.

2. MEASURING THE OSCILLATION OF A FUNCTION

2.1. Oscillation on an interval. $I \subseteq [a, b]$ an interval: $\text{Osc}(f, I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x)$.

Properties:

$$\text{Osc}(f, I) \geq 0$$

$$\text{Osc}(f, I) \geq |f(x) - f(y)|, \forall x, y \in I.$$

$$\text{Osc}(f, I) = \sup_{x, y \in I} |f(x) - f(y)| = \sup_{x, y \in I} f(x) - f(y).$$

$$\text{For } I = [c, d], J = [d, e], I \vee J = [c, e]: \max\{\text{Osc}(f, I), \text{Osc}(f, J)\} \leq \text{Osc}(f, I \vee J) \leq \text{Osc}(f, I) + \text{Osc}(f, J).$$

2.2. Oscillation at a point. $\text{Osc}(f, x) := \inf \text{Osc}(f, I)$, where the infimum is taken over all open intervals I containing x . Properties:

$$\text{Osc}(f, x) \geq 0, \forall x \in [a, b].$$

$$\text{Osc}(f, x) = 0 \text{ if and only if } f \text{ is continuous at } x.$$

3. PARTITIONS OF $[a, b]$

3.1. Definition. A partition P of $[a, b]$ is the datum of the following:

- an integer $n \geq 1$
- a set of division points $x_k, 0 \leq k \leq n$ such that: $a = x_0 < x_1 < \dots < x_n = b$
- a set of intermediate points $a_k, 1 \leq k \leq n$ such that: $a_k \in [x_{k-1}, x_k]$

$[x_{k-1}, x_k]$ are the partition intervals.

The norm of the partition: $\|P\| = \max_{1 \leq k \leq n} |x_k - x_{k-1}|$ (the length of the largest partition interval) measures how "fine" P is.

Example: for $n \geq 1$, P_n is the standard partition given by $x_k = a + \frac{k}{n}(b-a)$, $0 \leq k \leq n$, and $a_k = x_k$, $1 \leq k \leq n$. Then $\|P_n\| = \frac{b-a}{n}$.

3.2. Various sums associated to P .

$$S(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot f(a_k)$$

$$S^+(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot \sup_{[x_{k-1}, x_k]} f$$

$$S^-(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot \inf_{[x_{k-1}, x_k]} f$$

Note: the definition of $S^\pm(f, P)$ does not depend on the choice of intermediate points.

$$\text{Note: } S^-(f, P) \leq S(f, P) \leq S^+(f, P).$$

3.3. Lower and upper Riemann integrals.

$$S^-(f) = \sup_Q S^-(f, Q); \quad S^+(f) = \inf_Q S^+(f, Q)$$

3.4. Oscillation of f over P .

$\text{Osc}(f, P) := \sum_{k=1}^n (x_k - x_{k-1}) \cdot \text{Osc}(f, [x_{k-1}, x_k])$, the weighted sum of the oscillation of f over the partition intervals.

$$\text{Note: } \text{Osc}(f, P) = S^+(f, P) - S^-(f, P).$$

3.5. Key inequality.

3.6. If P and P' are two arbitrary partitions of $[a, b]$, we call $P \vee P'$ any partition of $[a, b]$ whose set of division points is the union of the corresponding sets of division points of P and P' (we don't care here about the intermediate points of $P \vee P'$). Then:

$$S^-(f, P') \leq S^-(f, P \vee P') \leq S^+(f, P \vee P') \leq S^+(f, P)$$

$$\text{Immediate consequence: } \begin{cases} S^-(f, P) \leq S^-(f) \leq S^+(f) \leq S^+(f, P) \\ S^+(f) - S^-(f) \leq \text{Osc}(f, P) \end{cases}$$

In particular $S^\pm(f)$ are (finite) real numbers.

4. RIEMANN INTEGRAL

4.1. **Definition.** f is Riemann integrable (RI) on $[a, b]$ if and only if there exists a real number $I \in \mathbb{R}$ with the following property:

Given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that: $|S(f, P) - I| \leq \epsilon$, whenever $\|P\| \leq \delta(\epsilon)$

In other words, $I = \lim_{\|P\| \rightarrow 0} S(f, P)$.

Notation: if f is RI, $I = \int_a^b f(x)dx = \int_a^b f dx = \int_a^b f$.

Example 1. The function $f : [a, b] \rightarrow \mathbb{R}$, $f(x) = C$ (constant), $\forall x$ is RI and $\int_a^b f = C(b - a)$.

Example 2. The function $f : [a, b] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not RI.

Proof. Let

R_n : the partition with $x_k = a + \frac{k}{n}(b - a)$ and $a_k \in [x_{k-1}, x_k] \cap \mathbb{Q}$

Q_n : same division points and intermediate points $b_k \in [x_{k-1}, x_k] - \mathbb{Q}$.

Then: $S(f, R_n) = (b - a)$, $S(f, Q_n) = 0$.

Yet $\|R_n\| = \|Q_n\| = \frac{b-a}{n} \rightarrow 0$ as $n \rightarrow \infty$. Therefore the limit $\lim_{\|P\| \rightarrow 0} S(f, P)$ does not exist.

4.2. **Equivalent formulation.** The following plays an analogous role to the theorem which characterizes convergent sequence by the Cauchy property.

Theorem 1. Assume $f : [a, b] \rightarrow \mathbb{R}$ is bounded. The following properties are equivalent:

- f is RI on $[a, b]$.
- $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that: $\text{Osc}(f, P) \leq \epsilon$ whenever $\|P\| \leq \delta(\epsilon)$.
- Given $\epsilon > 0$, there exists a partition Q such that $\text{Osc}(f, Q) \leq \epsilon$.
- $S^-(f) = S^+(f)$

Proof: $a \Leftrightarrow b$ is straightforward in view of the key inequality (previous section). $c \Leftrightarrow d$ is also straightforward in view of the same. $b \Rightarrow c$ is trivial. $c \Rightarrow b$ is a consequence of:

Lemma 6.2.1 [p. 221]. For a partition P , define $\nu(P) = \min_k |x_k - x_{k-1}|$, the length of the shortest partition interval. Then

$$\|Q\| \leq \nu(P) \Rightarrow \text{Osc}(f, Q) \leq 3 \text{Osc}(f, P)$$

4.3. **Continuous functions are RI.** A continuous function $f : [a, b] \rightarrow \mathbb{R}$ is uniformly continuous (theorem regarding continuous functions on compact domains).

This means: given $\epsilon > 0$, let $\delta(\epsilon)$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $|x - y| \leq \delta(\epsilon)$.

Let P an arbitrary partition such that $\|P\| \leq \delta(\epsilon)$.

This implies, for each k , $|x_k - x_{k-1}| \leq \delta(\epsilon)$, hence $\text{Osc}(f, [x_{k-1}, x_k]) \leq \epsilon$.

Summing up: $\text{Osc}(f, P) \leq \sum_k (x_k - x_{k-1}) \cdot \epsilon = \epsilon(b - a)$.

Conclusion: f is RI, in view of Theorem A.

4.4. General properties.

- $f : [a, b] \rightarrow \mathbb{R}$ is RI and $f(x) \geq 0, \forall x \in [a, b] \Rightarrow \int_a^b f(t)dt \geq 0$.
- $[c, d] \subseteq [a, b]$ and f is RI on $[a, b] \Rightarrow f$ is RI on $[c, d]$.
- f is RI on $[a, c]$ and $[c, b] \Rightarrow f$ is RI on $[a, b]$ and $\int_a^c f + \int_c^b f = \int_a^b f$.
- Translation-invariance: $\int_a^b f(x + c)dx = \int_{a+c}^{b+c} f(x)dx$.
- Scaling: $\int_a^b f(cx)dx = \frac{1}{c} \int_{ac}^{bc} f(x)dx$.
- Absolute value inequality: $f : [a, b] \rightarrow \mathbb{R}$ is RI $\Rightarrow |f| : [a, b] \rightarrow \mathbb{R}$ is RI (the converse is not true!) and $|\int_a^b f| \leq \int_a^b |f|$

4.5. **Fundamental theorem of calculus.** If $f : [a, b] \rightarrow \mathbb{R}$ is RI, then f is RI on each $[a, x]$, for $x \in [a, b]$.

Can define $F(x) = \int_a^x f(t)dt$. Then:

- $F : [a, b] \rightarrow \mathbb{R}$ is continuous, $F(a) = 0, F(b) = \int_a^b f(t)dt$ (homework problem)
- f continuous at some $x_0 \in (a, b) \Rightarrow F$ differentiable at x_0 and $F'(x_0) = f(x_0)$.

4.6. **Integral mean value.** f continuous (everywhere) $\Rightarrow \exists c \in (a, b)$ such that $\frac{1}{b-a} \int_a^b f(t)dt = f(c)$

5. POINTS OF CONTINUITY AND DISCONTINUITY

$\mathcal{C}(f) := \{x \in [a, b] : f \text{ is continuous at } x\}$

$\mathcal{D}(f) := \{x \in [a, b] : f \text{ is not continuous at } x\} = [a, b] - \mathcal{C}(f)$.

Recall: given $\alpha > 0$, the set $\{x : \text{Osc}(f, x) < \alpha\}$ is open.

Equivalent statement: $A(\alpha) := \{x \in [a, b] : \text{Osc}(f, x) \geq \alpha\}$ is closed.

Recall:

$\mathcal{C}(f) = \{x \in [a, b] : \text{Osc}(f, x) = 0\}$, $\mathcal{D}(f) = \{x \in [a, b] : \text{Osc}(f, x) > 0\}$

Consequently

$$\mathcal{C}(f) = \bigcup_{n \geq 1} \{x : \text{Osc}(f, x) < 1/n\}$$

$$\mathcal{D}(f) = \bigcup_{n \geq 1} \{x : \text{Osc}(f, x) \geq \frac{1}{n}\}$$

This shows that $\mathcal{C}(f)$ is a G_δ -set and $\mathcal{D}(f)$ is an F_σ -set.

Note: we will make use of the identity $\mathcal{D}(f) = \bigcup_{n \geq 1} A(1/n)$ later on.

6. SMALL SETS

6.1. Definition. For $I = (a, b)$ a bounded open interval of \mathbb{R} , let $|I| = b - a$.

In this section $E = \bigcup_{k=1}^N I_k$ denotes a finite union (not necessarily disjoint) open intervals.

Set $|E| = \sum_{k=1}^N |I_k|$.

Similarly $E' = \sum_{k=1}^{\infty} I_k$ denotes a countable union of open intervals.

Set $|E'| = \sum_{k=1}^{\infty} |I_k|$.

Given set $A \subseteq \mathbb{R}$, we have the following definitions:

- $m(A) = 0 \Leftrightarrow \inf_{A \subseteq E} |E| = 0$ [set of content zero]
- $\mu(A) = 0 \Leftrightarrow \inf_{A \subseteq E'} |E'| = 0$ [negligible set]

Note: given a set $A \subseteq \mathbb{R}$, the following statements are equivalent:

- $m(A) = 0$
- given $\epsilon > 0$, there exist finitely many intervals I_1, \dots, I_N such that $A \subseteq \bigcup_{1 \leq n \leq N} I_n$ and $\sum_{n=1}^N |I_n| \leq \epsilon$.

A similar characterization holds for negligible sets, the only difference is that we take into account countable coverings (with open intervals).

Notation: by abuse of notation, by $m(A) \neq 0$ we will mean that A is not a set of content zero.

6.2. Properties.

1. $m(A) = 0 \Rightarrow \mu(A) = 0$.
2. A finite $\Rightarrow m(A) = 0$.
3. A countable $\Rightarrow \mu(A) = 0$.
4. $A \subseteq B$ and $m(B) = 0 \Rightarrow m(A) = 0$.
5. $m(A_n) = 0, 1 \leq n \leq N \Rightarrow m(\bigcup_{n=1}^N A_n) = 0$
6. $\mu(A_n) = 0, \forall n \geq 1 \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = 0$
7. Under the assumption that A is compact: $m(A) = 0 \Leftrightarrow \mu(A) = 0$.

Note: the compactness assumption cannot be replaced by anything weaker, such as boundedness.

Example: $S = \mathbb{Q} \cap [0, 1]$ is bounded, yet $\mu(S) = 0$ and $m(S) \neq 0$ [why?].

8. Assume $A = \bigcup_{n=1}^{\infty} A_n$ is an F_σ -set, with A_n compact sets. Then the following statements are equivalent:

- $\mu(A) = 0$.
- $m(A_n) = 0, \forall n \geq 1$.

7. CHARACTERIZATION OF RIEMANN INTEGRABLE FUNCTIONS

7.1. Assume $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable and fix $\alpha > 0$.

Claim: the set $A(\alpha) := \{x \in [a, b] : \text{Osc}(f, x) \geq \alpha\}$ has content zero.

Proof. Take any $\epsilon > 0$. f is RI \Rightarrow there exists a partition P such that $\text{Osc}(f, P) < \epsilon$ (we know that there is an abundance of such partitions, but only one is enough). Say $a = x_0 < \dots < x_n = b$ are the division points of P .

Strategy: we are going to use the fact that $\text{Osc}(f, P) = \sum(\text{length} \times \text{oscillation})$ and the fact that the intervals that contain elements of $A(\alpha)$ have oscillation bounded away from zero, to conclude that their combined length is small.

Let K_1 the indices $k \in \{1, \dots, n\}$ such that $(x_{k-1}, x_k) \cap A(\alpha) \neq \emptyset$.

Key observation: if $k \in K_1$ then $\text{Osc}(f, [x_{k-1}, x_k]) \geq \alpha$. Therefore:

$$\epsilon \geq \text{Osc}(f, P) = \sum_{\text{all } k} \text{length} \times \text{oscillation} \geq \sum_{k \in K_1} (x_k - x_{k-1}) \text{Osc}(f, [x_{k-1}, x_k]) \geq \sum_{k \in K_1} (x_k - x_{k-1}) \cdot \alpha$$

Conclusion: $\sum_{k \in K_1} |x_k - x_{k-1}| \leq \epsilon/\alpha$ and $A(\alpha)$ (except perhaps finitely many points) is included in $\bigcup_{k \in K_1} (x_{k-1}, x_k)$. Making ϵ as small as we want (α is fixed!) we conclude that $m(A(\alpha)) = 0$.

7.2. Summing-up. Assume $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable. Then $\mathcal{D}(f) = \bigcup_{n=1}^{\infty} A(1/n)$ and we saw that $m(A(1/n)) = 0$ for all $n \geq 1$. This is equivalent to: $\mu(\mathcal{D}(f)) = 0$, *the set of points of discontinuity is negligible*. The converse of this fact is also true, but we will skip the proof (essentially chasing through the definitions). We have:

Theorem 2. Assume $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function. Then the following statements about f are equivalent:

- a) f is Riemann integrable on $[a, b]$.
- b) $\mu(\mathcal{D}(f)) = 0$.
- c) $m(\{x : \text{Osc}(f, x) \geq \alpha\}) = 0, \forall \alpha > 0$.

7.3. Corollary. Assume $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function, with a countable number of discontinuities. Then f is Riemann integrable.

7.4. Dirichlet's example. Dirichlet's function $f : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \gcd(p, q) = 1 \\ 0, & x \notin \mathbb{Q} \end{cases}$$

It has the property that $\mathcal{D}(f) = \mathbb{Q}$, countable. Therefore f is RI. By taking Riemann sums of partitions with irrational intermediate points we see that $\int_0^1 f(x)dx = 0$.

8. APPENDIX: MORE ON SETS OF CONTENT ZERO, CANTOR SET, ETC.

8.1. Note ¹: a set of content zero is automatically negligible.

8.2. While it is true that every finite set has content zero, the converse is not true. A simple example of an infinite set of content zero: $A = \{1, 1/2, 1/3, \dots\}$.

8.3. It is true that a countable set is negligible: if A is countable, then $\mu(A) = 0$.

Proof: assume $A = \{x_1, x_2, \dots\}$. Then for any $\epsilon > 0$, A is a subset of $\bigcup_{n=1}^{\infty} I_n$, where $I_n = (x_n - \frac{\epsilon}{2^n}, x_n + \frac{\epsilon}{2^n})$, and $\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$.

8.4. The converse of the previous statement is not true. The Cantor set is the typical example of an uncountable set which has content zero (implicitly negligible).

Proof: let S the Cantor set defined on p. 96 [textbook]. S is obtained by removing an interval of length $1/3$, two intervals of length $1/3^2$, four intervals of length $1/3^3$, etc... After n steps we removed intervals of total length:

$$\sum_{k=1}^n \frac{2^{k-1}}{3^k} = \frac{1}{2} \sum_{k=1}^n \left(\frac{2}{3}\right)^k = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}} = 1 - \left(\frac{2}{3}\right)^n$$

Therefore S is covered by a finite union of intervals (the complement of those removed) of combined length $1 - (1 - (\frac{2}{3})^n) = (\frac{2}{3})^n$. As $\lim_{n \rightarrow \infty} (\frac{2}{3})^n = 0$ it shows that $m(S) = 0$.

Conclusion: S is compact, uncountable, of content zero.

8.5. It is true that a set of negligible set has empty interior.

Proof: assume $m(A) = 0$ and $\text{Int}(A) \neq \emptyset$. Then there exists $(a, b) \subseteq A$. It is not hard to prove that $A \subseteq \bigcup_{n=1}^N I_n \Rightarrow \sum_{n=1}^N |I_n| \geq (b - a)$, thus contradicting the assumption that $m(A) = 0$.

¹Most of this section is not exam or homework material, it is merely to answer certain questions that might occur while reading about Riemann integrability

8.6. The converse of the previous statement is not true. Example: $T = [0, 1] - \mathbb{Q}$ has empty interior (since the rationals are dense) yet is not negligible.

Proof: assume T is negligible. Since the complement $T' = [0, 1] \cap \mathbb{Q}$ is negligible (countable) as well, it follows that the union $T \cup T' = [0, 1]$ is negligible as well, contradiction.

8.7. It is true that a set A of content zero is nowhere dense, in the sense that $\text{Int}(\overline{A}) = \emptyset$.

Proof: $m(A) = 0 \Rightarrow \mu(A) = 0$ (why?). Hence \overline{A} has empty interior.

Note: we already saw that a set of content zero A has empty interior. Now we see that even the possible larger \overline{A} has empty interior. This property does not hold for negligible sets, which are not necessarily that small.

8.8. The converse of the previous statement is not true. In fact, there exists a set W such that $\text{Int}(\overline{W}) = \emptyset$ and yet W is not even negligible.

Example: let $q_1, q_2, \dots, q_n \dots$ an enumeration of $[0, 1] \cap \mathbb{Q}$. Set $W := [0, 1] - \bigcup_{n \geq 1} (q_n - \frac{1}{3^n}, q_n + \frac{1}{3^n})$. Then

- W is non-negligible [reason: we took away a countable union of intervals of combined length < 1]
- W is nowhere dense: $\text{Int}(\overline{W}) = \emptyset$ [reason: we took away the rationals]

8.9. It is not true that a negligible set is nowhere dense. Example: $T' = \mathbb{Q} \cap [0, 1]$ is negligible (countable) and yet $\text{Int}(\overline{T'}) = (0, 1) \neq \emptyset$.