PRACTICE PROBLEMS: SEQUENCES

1. Prove or disprove: there exists a monotone sequence of real numbers $(x_n)_{n\geq 1}$ such that for any given $q \in \mathbb{Q}$, there exists $n \geq 1$ such that $q = x_n$. What happens if we drop the condition of monotonicity?

2. Assume $\{x_n\}_{n\geq 1}$ is a Cauchy sequence and $f : \mathbb{N} \to \mathbb{N}$ an injective sequence.
a) True/false: $\{x_{f(n)}\}_{n\geq 1}$ is a subsequence of $\{x_n\}_{n\geq 1}$.
b) True/false: $\{x_{f(n)}\}_{n\geq 1}$ is a Cauchy sequence.

3. Find $\limsup x_n$ and $\liminf x_n$, where:
a) $x_n = \frac{(-1)^n}{n^2}$; b) $x_n = 1 + (-1)^n$, for $n \geq 1$.

4. Prove that a bounded monotone sequence is convergent.

5. Assume x_n is a sequence with positive terms. Prove that either $\lim_{n\to+\infty} x_n \to +\infty$ or x_n has a bounded subsequence.

6. Prove that a positive sequence is either bounded or it has a subsequence that tends to $+\infty$.

7. Assume x_n is a monotone increasing, unbounded sequence. Prove that $\lim_{n\to+\infty} x_n = +\infty$.

8. True or false: if x_n is convergent, then x_n is bounded.

9. Assume a_n, $n \geq 1$ is a sequence of rational numbers such that $\lim a_n = \sqrt{2}$. Let $a_n = \frac{p_n}{q_n}$, with p_n, q_n integers, and $q_n > 0$. Prove that $\lim q_n \to +\infty$.

10. Prove that $\lim_{n\to\infty} (\frac{2}{3})^n = 0$.

11. Assume x_n is a sequence such that $x_1 = -1$ and $|x_{n+1} - x_n| \leq \frac{1}{2^n}$ $\forall n \geq 10$. a) Prove that x_n is convergent. b) What happens if we replace $\frac{1}{2^n}$ by $\frac{1}{n^2}$? c) What happens if we replace $\frac{1}{2^n}$ by $\frac{1}{n}$?

12. Assume $x_n \to 1$. Prove that $\limsup_n x_n y_n = \limsup_n y_n$, for any sequence y_n.

13. True or false: if x_n are y_n are bounded sequences, then $\limsup x_n y_n = \limsup x_n \cdot \limsup_n y_n$.

14. Assume $x_n \to L > 0$. a) Prove that $\frac{1}{n}(x_1 + x_2 + \cdots + x_n) \to L$.
b) Prove that $(x_1 x_2 \ldots x_n)^{1/n} \to L$.

15. Assume $\lim x_{n+1}/x_n = L > 0$. Prove that $x_n^{1/n} \to L$.

16. Compute $\lim_{n\to+\infty} n^{1/n}$ and $\lim_{n\to+\infty} (n!)^{1/n}$.