PRACTICE PROBLEMS: SETS

1. Let A a finite set with n elements. Prove that the power set $\mathcal{P}(A)$ has 2^n elements.

2. Give a complete proof that $\mathfrak{c}^2 = \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of $\mathbb{R}$.

3. a) Prove that there exists a positive real number a such that $a^2 = 3$. b) Prove that $a \notin \mathbb{Q}$.

4. Prove the existence of a bijective function $f : (0,1) \times (0,1) \to (0,1)$.

5. Let A a subset of $\mathbb{R}$. Define $\text{int}(A)$, the interior of A. What is $\text{int}(\mathbb{Q})$?

6. True/false: for a subset A of $\mathbb{R}$, $\overline{A} - A \subset A^d$ (the derived set).

7. Prove that if G is open and E is arbitrary then $G \cdot E := \{xy | x \in G, y \in E\}$ is open.

8. Prove that if A and B are compact sets, then $A + B := \{x + y | x \in A, y \in B\}$ is also compact.

9. Argue that $\mathbb{Q}$ is dense in $\mathbb{R}$.

10. Prove that an arbitrary intersection of compact subsets of $\mathbb{R}$ is a compact subset of $\mathbb{R}$.

11. Prove that if B is a bounded subset of $\mathbb{R}$, then $\overline{B}$ is compact.

12. Prove that a finite union of compact sets is a compact set:
a) by using the characterization of compacts subsets of $\mathbb{R}$;
b) by using the open covering property of compact sets.

13. True or false: an uncountable intersection of compact sets is compact.

14. Prove that the intervals are the only connected subsets of $\mathbb{R}$.

15. Prove that $\mathbb{R}$ and $\emptyset$ are the only subsets of $\mathbb{R}$ which are both closed and open.

16. Let $A \subseteq \mathbb{R}$ a subset of the real numbers. Prove that every open cover of A has a countable subcover.

17. Assume $C \subseteq \mathbb{R}$ is bounded. Prove that $\sup(C) \in \overline{C}$.

18. Assume $C \subseteq \mathbb{R}$ is compact. Prove that $\sup(C) \in C$.

19. Prove that $\mathbb{N} := \{1, 2, \dots\}$ is a countable union of compact subsets.

20. Assume A is a set of real numbers which does not contain any of its limit points, i.e. such that $A \cap A^d = \emptyset$.
a) Prove there exists a bijective function $h : \mathbb{Z} \to A$
b) Does there exist a monotone bijective function $h : \mathbb{Z} \to A$?

21. Prove that if A is of content zero, then the closure $\overline{A}$ is of content zero.

22. Assume $E \subseteq [0,1]$ is compact. Prove that E is negligible if and only if E is of content zero.

23. Prove that $[0,1] - \mathbb{Q}$ is not negligible.

24. Prove that $\mathbb{R} - \mathbb{Q}$ is a G_δ-set.