LANDAU NOTATION AND DERIVATIVE

1. Landau Notation

1.1. Assume $P(x)$ a statement that depends on the variable x.
”As $x \to x_0$, eventually $P(x)$” means that there exists an open neighborhood V of x_0 such that $P(x)$ is true for every $x \in V$.
”As $n \to \infty$, eventually $P(n)$” means that $\exists N > 0$ such that for $n > N$, $P(n)$ is true.
Example: $\lim_{x \to x_0} f(x) = L$ is the statement that $\forall \epsilon > 0$, eventually $|f(x) - L| < \epsilon$.
In this case ”eventually” means ”$\exists \delta > 0$ (possibly depending on ϵ, so we write $\delta = \delta(\epsilon)$) such that for $|x - x_0| < \delta(\epsilon)$.”

1.2. Assume $f(x)$ and $g(x)$ are two functions defined in a neighborhood of x_0. Moreover, let's assume that $g(x)$ is reasonably simple, while $f(x)$ is a function whose behavior (near x_0) we want to understand in terms of that of $g(x)$.

1.3. **Definition.** ”As $x \to x_0$, $f(x) = O(g(x))$” means that there exists $C > 0$ such that eventually $|f(x)| \le C|g(x)|$, as $x \to x_0$.

Equivalent meaning: there exists an open neighborhood of x_0 where f (in absolute value) is no bigger than g, up to a constant multiple.

Equivalent definition: $\exists C > 0$ and $\delta > 0$ such that $|f(x)| \le C\,|g(x)|$ for $|x - x_0| < \delta$.

Note: this is almost the same as saying that the quotient $f(x)/g(x)$ is eventually bounded as $x \to x_0$, the only problem with this definition being that $g(x)$ might actually vanish at x_0 (or in a neighborhood of x_0), in which case the quotient $f(x_0)/g(x_0)$ is not well-defined.

Remark: with this definition, if $g(x_0) = 0$ and $f(x) = O(g(x))$ as $x \to x_0$, then $f(x_0) = 0$.

1.4. **Definition.** ”As $x \to x_0$, $f(x) = o(g(x))$” means $\forall\, \epsilon > 0$, eventually $|f(x)| \le \epsilon \cdot |g(x)|$ as $x \to x_0$.

Equivalent meaning: f is smaller (in absolute value) than any given fraction of g in a small enough neighborhood of x_0.

Equivalent definition: $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $|f(x)| \le \epsilon \cdot |g(x)|$ for $|x - x_0| < \delta(\epsilon)$.

Note: this is almost the same as saying that $f(x)/g(x) \to 0$ as $x \to x_0$, the problem with this formulation being once more that $g(x)$ might actually vanish at x_0 (or in a neighborhood of x_0).

Remark: assume $g(x_0) = 0$ but $g \neq 0$ on $V - \{x_0\}$, where V is some neighborhood of x_0. Then $f(x) = o(g(x))$ if and only if $f(x_0) = 0$ **and** $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

1.5. **Definition.** ”As $x \to +\infty$, $f(x) = O(g(x))$” means that there exists $C > 0$ and $N > 0$ such that $|f(x)| \le C|g(x)|$ for all $x > N$.

1.6. **Definition.** ”As $x \to +\infty$, $f(x) = o(g(x))$” means that $\forall \epsilon > 0$, $\exists N(\epsilon) > 0$ such that $|f(x) \le \epsilon \cdot |g(x)|$ for $x > N(\epsilon)$.

1.7. **Examples.**

- f is continuous at x_0 if and only if $f(x) = f(x_0) + o(1)$ as $x \to x_0$.
- If $f(x) = o(g(x))$ as $x \to x_0$, then $f(x) = O(g(x))$ as $x \to x_0$
- $x - x_0 = o(1)$ as $x \to x_0$
- $(x - x_0)^2 = o(x - x_0)$
- If $k < N$, $x^k = o(x^N)$ as $x \to +\infty$
- If $k < N$, $x^N = o(x^k)$ as $x \to 0$
- For any $a > 0$, $\ln x = o(x^a)$ as $x \to +\infty$
- $x = O(\sqrt{x})$ as $x \to 0$
- $(x - x_0)^2 = o(x - x_0)$ as $x \to x_0$
- $x - 1 = O((x-1)^2)$ as $x \to 0$
- $\sin(x) = O(x)$ as $x \to 0$, and $\sin x \neq o(x)$ as $x \to 0$.
- $\sin(x) = x - \frac{x^3}{6} + O(x^5)$ as $x \to 0$

2. Derivative

2.1. **Set-up.** $f(x)$ is a real-valued function defined on some neighborhood of a.

2.2. **Definition.** We say that f is differentiable at a provided the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists (and is finite). If that is the case, we denote the limit by $f'(a)$.

Note: by definition, a necessary condition for f to be differentiable at a is that f is continuous at a.

2.3. If f is differentiable at a, we can then re-write the limit as

$$\lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}=0$$

which is the same as

$$f(x)=f(a)+f'(a)(x-x_0)+o(x-a)$$

2.4. **Proposition.** The following statements are true:
i) f is differentiable at a.
ii) There exists a real number $k>0$ such that $f(x)=f(a)+k(x-a)+o(x-a)$ as $x\to a$.

Proof. $i\Rightarrow ii$. Take $k=f'(a)$.
$ii\Rightarrow i$. Assume $f(x)=f(a)+k(x-a)+o(x-a)$ as $x\to a$. This simply means that $f(x)-f(a)-k(x-a)=o(x-a)$, which is equivalent to

$$\lim_{x\to a}\frac{f(x)-f(a)-k(x-a)}{x-a}=0$$

But we can re-write the limit as

$$\lim_{x\to a}\Big(\frac{f(x)-f(a)}{x-a}-k\Big)=0\Leftrightarrow\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=k$$

Thus f is differentiable at a and $f'(a)=k$.

Proposition. If f is differentiable at a, then $f(x)$ is continuous at a.

First proof: $\lim_{x\to a,x\neq a}f(x)-f(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)=k\cdot 0=0$.
Second proof: $f(x)=f(a)+f'(a)(x-a)+o(x-a)=f(a)+O(x-a)$, hence $\lim_{x\to a}f(x)=f(a)$.

2.5. **Linear approximation.** Assume f is continuous at a. For an arbitrary $k\in\mathbb{R}$, define $T_k(x)=f(a)+k(x-a)$. Then $T_k(x)$ is a linear function which takes the same value at a as f: $T_k(a)=f(a)$ (draw the graph of T_k, a line). Since both T_k and f are continuous at a, we can write $f(x)=T_k(x)+o(1)$ as $x\to x_0$. We might want to conclude that T_k provides a good enough approximation for f near a (after all, the difference $f-T_k$ does go to 0 as $x\to a$). However, if we draw the graphs of f and T_k (with k chosen "arbitrary") we see that aside from passing through the same point, the two curves don't have much else in common. So the question is: is there a certain k for which the difference $f(x)-T_k(x)$ (as $x\to a$) is even smaller than $o(1)$?
The first natural "quantity" smaller than $o(1)$ is $O(x-a)$, but that is not good enough (let's not wonder why). The next candidate would be $o(x-a)$. Does there exists a k such that the corresponding T_k approximates f to $o(x-a)$ as $x\to a$?

From the discussion of the previous section we see that the answer is yes, provided f is differentiable at a. If such a k exists, then it is unique: $k=f'(a)$. Moreover, the graph of the corresponding T_k is the tangent line at a to the graph of f.
Remark: we now have the equation of the tangent line of a function. Being the graph of T_k with $k=f'(a)$, it has equation $Y=f'(a)(X-a)$.