OCCURENCES OF SUP AND INF IN ANALYSIS I

The following theorem is one of the main themes of this course:

Theorem 1. $\mathbb{R}$ is complete.

In other words, any Cauchy sequence of real numbers is convergent.
The next theorem was a consequence of the completeness of $\mathbb{R}$:

Theorem 2. Assume the set $E \subseteq \mathbb{R}$ is bounded from above. Then E admits a least upper bound.

Comment. We denote this by $l.u.b(E)$ or $\sup(E)$. By definition, it has the following properties:

- $\sup(E) \in \mathbb{R}$
- $x \leq \sup(E)$, $\forall x \in \sup(E)$
- $\forall \epsilon > 0$, $\sup(E) - \epsilon$ is *not* an upper bound

The last condition implies that $\forall \epsilon > 0$, $\exists z \in E$ (possibly depending on ϵ) such that $z > \sup(E) - \epsilon$. In particular, this shows that $\sup E \in \overline{E}$.

Remark. The existence of the least upper bound of a bounded set is as strong as the completeness of $\mathbb{R}$. It is so fundamental that certain textbooks state it as an axiom and derive from it the completeness of $\mathbb{R}$. It should not be surprising that the least upper bound theorem is at the heart of some of the important theorems that we encountered in this class. Here is a list a series of such theorems.

1. Proposition. There exists a positive real number $x > 0$ such that $x^2 = 3$.

Proof. Consider the following set $E := \{x \geq 0, x^2 \leq 3\}$. Then $M = 2$ is certainly an upper bound for E [why?]. So E is bounded from above. Consider $y = \sup(E) \in \mathbb{R}$. We will prove that $y^2 = 3$. y has the properties: a) $x \leq y$, $\forall x \in E$; b) $\forall n \geq 1$, $\exists x_n \in E$ such that $x_n > y - \frac{1}{n}$. Combining a) and b) we have $x_n \leq y \leq x_n + \frac{1}{n} \Rightarrow |x_n - y| \leq \frac{1}{n}$. This implies $y = \lim x_n$ and hence $y^2 = \lim x_n^2$. Since $x_n^2 \leq 3$, for all $n \geq 1$, we then have $y^2 \leq 3$.
Assume $y^2 < 3$. We will reach a contradiction.
First method. We will produce a real number y' such that $y' > y$ and $y'^2 \leq 3$. This will put $y' \in E$ and the fact that $y' > y$ then contradicts the fact that y is an upper bound for E. Since we need $y' > y$, let $y' = y + a$, and lets determine $a > 0$. What we need is $y'^2 = (y+a)^2 = y^2 + 2ay + a^2 < 3$. Thus it will be enough if both $2ay$ and a^2 are smaller than ϵ, where $h := \frac{3-y^2}{2}$. In other words what we need is $a < \frac{h}{2y}$ and $a^2 < h$. If $h \leq 1$ it is enough to take $a = \min\{\frac{h}{2y}, h\}$. If $h > 1$ it is enough to take $a = \min\{\frac{h}{2y}, 1\}$.
Second method. Clearly $\lim(y + \frac{1}{n})^2 = y^2$. Set $\epsilon = \frac{3-y^2}{2} > 0$. Choose any $n \geq N(\epsilon)$. Then $y' = y + \frac{1}{n}$ has the property that $y' > y$ and $y'^2 < 3$.

2. Proposition. A monotone & bounded sequence is convergent.

Proof. Assume x_n is bounded and monotone increasing: $m \leq x_n \leq x_{n+1} \leq M$, $\forall n \geq 1$, where m and M are (some) lower and upper bounds. The existence of the lower bound m is irrelevant in this case since an increasing sequence is automatically bounded from below. The existence of M tells us that the set $E = \{x_n : n \geq 1\}$ is bounded from above, hence it has a least upper bound. Let y be this least upper bound. This is a real number with the properties: a) $y \geq x_n$, $\forall n \geq 1$; b) given $\epsilon > 0$, there exists $N \geq 1$ (depending on ϵ) such that $x_N \geq y - \epsilon$. Combine a) and b): since x_n is increasing, it means that for $n \geq N$ we in fact have $y - \epsilon \leq x_n \leq y \Rightarrow |x_n - y| \leq \epsilon$. This shows that $\lim x_n = y$, so the sequence is convergent.

Note: a monotone sequence always has a limit (possibly infinite): if x_n is monotone, then

$$\lim x_n = \begin{cases} \sup x_n, & \text{if increasing} \\ \inf x_n, & \text{if decreasing} \end{cases}$$

3. Distinguished limit points of a given sequence.

Given an arbitrary (say bounded, for simplicity) sequence x_n, we can create two distinguished limit points of the sequence by taking (repeatedly) the sup and inf of various sets. Let $z_n := \sup_{k \geq n} x_k$, for $n \geq 1$. Then z_n is a decreasing sequence and $\lim z_n = \inf z_n =: \limsup x_n$ is a limit point for x_n.

Note: x_n is convergent if and only if $\limsup x_n = \liminf x_n$, in which case $\lim x_n$ equals these. Therefore, if x_n is convergent to begin with, we have a (more or less) constructive procedure for determining the limit of x_n.

4. Lower and Upper Riemann Integrals.

Given a bounded function $f : [a, b] \to \mathbb{R}$, integrable or not, one can always construct the following quantities: $\int^+ f = \inf_P S^+(F, P)$ and $\int^- f = \sup_P S^-(f, P)$ where the inf and sup are taken over all partitions P of f. The fact that quantities are finite is proved by noticing a certain order that exists among the Riemann sums associated to (two) different partitions (and their mutual refinement).
Recall; $f : [a, b]$ is integrable if and only if $\int^+ f = \int^- f$.

5. If E is compact, then $\sup(E), \inf(E) \in E$.

This has two do with a fact alluded before, namely that if E is a bounded set, then $\sup(E), \inf(E) \in \overline{E}$ (the closure of E). If E is compact then E is closed and this means $\overline{E} = E$. Applications:
a) continuous functions on compact domains attain their maximum and minimum. This combines the previous statement with the fact that a continuous function maps compact sets into compact sets.
b) Mean value theorem. The proof of MVT involves statement a).