Error on Ex. 15/p.232

It is not true that $G_{\delta}$-sets which are dense must be the entire $\mathbb{R}$. Example: $\mathbb{R} - \mathbb{Z}$ is open (hence trivially $G_{\delta}$) and dense in $\mathbb{R}$.

I suggest you think about the following problem:

15. Prove that a dense $G_{\delta}$ set is uncountable.

In particular, this shows that $\mathbb{Q}$ is not a $G_{\delta}$. In particular, this shows that one cannot construct a function $f : [0, 1] \rightarrow \mathbb{R}$ which is continuous only at the rational points.

If you find 1. hard, you might want to try this:

15. Prove that $\mathbb{Q} \cap [0, 1]$ is not a $G_{\delta}$ set, but $\mathbb{R} - \mathbb{Q}$ is.