10. THE DUAL OF $SU(2)$. SPECIAL FUNCTIONS ON $\mathbb{S}^3$

1. Preliminaries

1.1. **Lemma.** Assume (π, V) is a representation of the connected Lie group G and π_* is the associated action of the Lie algebra $\mathfrak{g}$. Then V is G-irreducible if and only if it is $\mathfrak{g}$-irreducible.

1.1.1. *Proof.* "$\Leftarrow$". Assume V is $\mathfrak{g}$-irreducible. Let $W \subseteq V$ a non-zero G-invariant subspace. We need to prove that $W = V$. For $X \in \mathfrak{g}$ and $w \in W$ we have

$$\pi_*(X)w = \lim_{t \to 0} \frac{\pi(\exp(tX))w - w}{t} \in W$$

hence W is $\mathfrak{g}$-invariant as well. Since V is $\mathfrak{g}$-irreducible it follows that $W = V$.

"$\Rightarrow$". Assume V is G-irreducible. Let $W \subset V$ a non-zero $\mathfrak{g}$-invariant subspace. For $g = \exp(X)$ and $w \in W$

$$\pi(g)w = \exp(\pi_*(X))w = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!} \pi_*(X)^k w \in W$$

Hence W is $\pi(\exp \mathfrak{g})$-invariant. Since G is connected, the set $\exp(\mathfrak{g})$ generates G, and hence W is also G-invariant. But then $W = V$.

1.2. **Complexified Lie algebra.** Fact: $su(2) \otimes_{\mathbb{R}} \mathbb{C} = sl(2, \mathbb{C})$. What this means is that every element $X \in sl(2, \mathbb{C})$ can be written (uniquely) as $X = A + iB$, where $A, B \in su(2)$. It can be easily verified that $A = \frac{1}{2}(X - \overline{X}^t)$ and $B = \frac{1}{2i}(X + \overline{X}^t)$.

Assume now that $\mathfrak{h}$ is a Lie algebra over $\mathbb{C}$ and $\alpha : su(2) \to \mathfrak{h}$ is a homomorphism of *real* Lie algebras. In other words, $\alpha \in Hom_{\mathbb{R}}(su(2), \mathfrak{h})$ is an $\mathbb{R}$-linear map that preserves bracket relations. Then α can be naturally extended extended to a homomorphism of *complex* Lie algebras

$$\tilde{\alpha} : sl(2, \mathbb{C}) \to \mathfrak{h}, \quad \tilde{\alpha}(X) = \alpha(\frac{X - \overline{X}^t}{2}) + i\alpha(\frac{X + \overline{X}^t}{2i})$$

Note that $\tilde{\alpha}(X) = \alpha(X)$ if $X \in su(2)$. We will use the same later α to denote $\tilde{\alpha}$ as well. if there is no danger of confusion.

1.2.1. *Remark.* Assume V (a complex vector space) is an $su(2)$-module. This means that there is a homomorphism of real Lie algebras $\phi : su(2) \to gl(V)$. Since $gl(V)$ is naturally a complex Lie algebra, this homomorphism extends to $\phi : sl(2, \mathbb{C}) \to gl(V)$. In other words, V is an $sl(2, \mathbb{C})$-module. It is immediate to check that V is $su(2)$-irreducible if and only if it is $sl(2, \mathbb{C})$-irreducible.

1.3. **Strategy to determine $\widehat{SU(2)}$.** Assume $(\pi, V) \in \widehat{SU(2)}$. Then V is an irreducible $sl(2, \mathbb{C})$-module. Hence we will determine $\widehat{SU(2)}$ in two steps as follows:

- Determine the irreducible $sl(2, \mathbb{C})$-modules
- Determine which of the above arise from actual representations of $SU(2)$

1.4. **Weights.** We will consistently use the following notation

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in sl(2, \mathbb{C}), \quad \tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SU(2)$$

Assume (π, V) is a unitary representation of $SU(2)$. Under the action of the torus

$$T := \left\{ k_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} : \theta \in \mathbb{R} \right\} \subset SU(2)$$

we have the following decomposition [homework]

$$V = \oplus_{m \in \mathbb{Z}} V(m), \quad V(m) := \{ v \in V : \pi(k_\theta)v = e^{im\theta}v, \forall k_\theta \in T \}$$

We say that $V(m)$ is the subspace of weight m, $m \in \mathbb{Z}$.

1.5. **Proposition.** $V(m) = \{ v \in V : \pi_*(H)v = mv \}$.

Proof: $\pi_*(H)v = \frac{1}{i}\pi_*(iH)v = \frac{1}{i}\frac{d}{dt}\pi(e^{itH})v|_{t=0} = \frac{1}{i}\frac{d}{dt}e^{imt}v|_{t=0} = mv.$

1.6. **Theorem.** We have

$$\pi_*(E_+) : V(m) \to V(m+2), \quad \pi_*(E_-) : V(m) \to V(m-2), \quad \pi(\tau) : V(m) \simeq V(-m)$$

1.6.1. *Proof.* Assume $v \in V(m)$. We have

$$\pi_*(H)\big(\pi(\tau)v\big) = \pi(\tau)\pi(\tau^{-1})\pi_*(H)\pi(\tau)v = \pi(\tau)\pi_*\big(\operatorname{Ad}(\tau^{-1})H\big)\ v$$

Since $\operatorname{Ad}(\tau^{-1})H = -H$ [check!], we have $\pi_*(H)(\pi(\tau)v) = \pi(\tau)(-mv) = -m\pi(\tau)v$, hence $\pi(\tau)v \in V(-m)$. Note that since $\pi(\tau)$ is invertible, $\pi(\tau) : V(m) \simeq V(-m)$ is an isomorphism.

To prove that E_+ is a *raising operator* we use the bracket relations

$$\pi_*(2E_+) = \pi_*([H, E_+]) = [\pi_*(H), \pi_*(E_+)] = \pi_*(H)\pi_*(E_+) - \pi_*(E_+)\pi_*(H)$$

Applying both sides to the vector $v \in V(m)$ we obtain

$$\pi_*(H)\Big(\pi_*(E_+)v\Big) = 2\pi_*(E_+)v + \pi_*(E_+)\pi_*(H)v = (m+2)\pi_*(E_+)v \Rightarrow \pi_*(E_+)v \in V(m+2)$$

Similarly for E_-.

2. IRREDUCIBLE $sl(2,\mathbb{C})$-MODULES

2.1. **Highest weight.** Assume now that V is irreducible under the action of $sl(2,\mathbb{C})$. Let n be the largest positive integer such that $V(n) \neq 0$. We say that n is the *highest weight* of V. Since $\pi(\tau) : V(n) \simeq V(-n)$ it follows that $V(-n)$ is the lowest weight, and

$$V = \sum_{-n \le k \le n} V(k)$$

with some $V(k)$ possibly zero. Let $W = \sum_{k \equiv n (\mathrm{mod}\, 2)} V(k)$. Then W is stable under the actions of H, E_+, E_-. Since these generate $sl(2,\mathbb{C})$, it means that W is a non-zero $sl(2,\mathbb{C})$-submodule, hence $V = W$. Therefore only the weights k, with $k \equiv n(\mathrm{mod}\, 2)$ and $|k| \le n$ are potential contributors. So we re-write:

$$V = \bigoplus_{|k| \le n,\ k \equiv n(\mathrm{mod}\, 2)} V(k)$$

Let k as above. We claim that $V(k) \neq 0$, so all such k contribute. For assume k_0 is such that $V(k_0) = 0$. Then $V_1 = \oplus_{k > k_1} V(k)$ is a proper invariant $sl(2,\mathbb{C})$-submodule, impossible. We will modify slightly the previous definition and call k a weight only if $V(k) \neq 0$.

2.2. **Generators.** Let $e_n \in V(n)$ a non-zero vector of highest weight. Define

$$e_{n-2} = E_- e_n \in V(n-2), \quad e_k = E_-^{(n-k)/2} e_n \in V(k)$$

for k a weight.

2.3. **Lemma.** $E_+ e_k = a_n(k)\ e_{k+2}$, where

$$a_n(k) = \frac{(n-k)(n-k+2)}{4}$$

*P*roof. By induction, since $a(n) = 0$ and $a(k) = a(k+2) + k + 2$.

2.4. Let e_n as above. Then $\bigoplus_k \mathbb{C}e_k$ is a $sl(2,\mathbb{C})$-submodule of V. Since V is assumed irreducible, we have

$$V = \oplus_{|k| \le n,\ k \equiv n(\mathrm{mod}\, 2)} \mathbb{C}e_k$$

where e_n is a non-zero vector of highest weight and e_k are obtained from e_n by successive applications of E_-.

2.4.1. In view of the above lemma, all $e_k \neq 0$, for k as above. For assume there exists k such that $e_k = 0$. Then $a_n(k+2)e_{k+2} = E_+ e_k = 0$. However $a_n(k) \neq 0$ for $-n \le k \le n-2$, hence $e_{k+2} = 0$. Iterating this argument we arrive at $e_n = 0$, contradiction. In particular $\dim V = n+1$.

2.5. **Classification.** Since the constants $a_n(k)$ do not depend on the choice of e_n, it follows that $sl(2,\mathbb{C})$ has a unique (up to equivalence) irreducible module of highest weight n, for any $n \ge 0$. We call this V_n.

3. Irreducible $SU(2)$ representations

3.1. **Recall.** The passage from $SU(2)$-representations to $sl(2,\mathbb{C})$-modules ensured that

$$\widehat{SU(2)} \subseteq \{V_n : n \geq 0\}$$

For each $n \geq 0$, we will construct a representation of $SU(2)$ which is $sl(2,\mathbb{C})$-equivalent to V_n. This will show that the above is in fact an equality.

3.2. **Homogeneous polynomials.** Let $\mathcal{H}_n = \{P(X,Y) = \sum_{k=0}^{n} c_k X^k Y^{n-k} : \; c_k \in \mathbb{C}\}$ the vector space of homogeneous polynomials of two variables, of degree n. Consider the following action of $SU(2)$ on $\mathcal{H}_n$ $(g,P) \mapsto \sigma_n(g)P$ given by

$$\sigma_n(g)P(X,Y) = P\big((X,Y)g\big) = P(aX+cY, bX+dY), \quad g = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in SU(2)$$

It is immediate to check that σ_n is a representation of $SU(2)$, in that $\sigma_n(gh) = \sigma_n(g)\sigma_n(h)$.

3.3. **Proposition.** For $n \geq 0$, V_n and $\mathcal{H}_n$ are equivalent $sl(2,\mathbb{C})$-modules.

3.3.1. *Proof.* Since $\sigma_n(H)(X^kY^{n-k}) = (2k-n)X^kY^{n-k}$, $\mathcal{H}_n(2k-n) = \mathbb{C}\; X^kY^{n-k}$. In particular $\mathcal{H}_n(k) \neq 0$ if and only if $|k| \leq n$ and $n \equiv n (\operatorname{mod} 2)$, and every weight space has dimension 1. The highest weight is n, and $\mathcal{H}_n(n) = \mathbb{C}\; X^n$. From the classification of irreducible $sl(2,\mathbb{C})$-modules it follows that $\mathcal{H}_n \simeq V_n$ as $sl(2,\mathbb{C})$-modules. This also says that each σ_n is an irreducible $SU(2)$ representation. We sum up with the following

3.4. **Theorem.** $\widehat{SU(2)} = \{\sigma_n : n \geq 0\}$. σ_n is the unique representation of highest weight n, and $\dim \sigma_n = n+1$. The trace of σ_n is given by

$$\chi_n(k_\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}\;, \quad k_\theta = \left[\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array}\right]$$

3.4.1. *Proof.* Since $\sigma_n(k_\theta)(X^jY^{n-j}) = e^{i(2j-n)}X^jY^{n-j}$ we have

$$\chi_n(k_\theta) = \operatorname{trace}\sigma_n(k_\theta) = \sum_{j=0}^{n} e^{i(2j-n)\theta} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(n+1)\theta}{\sin\theta}$$

The character χ_n is determined by its restriction to the maximal torus, since χ_n is a class function and every element of $SU(2)$ has a conjugate in T.

3.5. **Unitary structure.** We know that σ_n admits a unique (up to scalar multiplication) $SU(2)$-invariant hermitian structure. To determine it explicitly, we may assume that it is normalized so that

$$\|X^n\| = 1$$

We have the following relations [homework]

$$E_-(X^kY^{n-k}) = kX^{k-1}Y^{n-k+1}, \quad E_+(X^kY^{n-k}) = (n-k)X^{k+1}Y^{n-k-1}$$

Therefore

$$\begin{gathered}\big\langle E_+(X^kY^{n-k}), E_+(X^kY^{n-k})\big\rangle = \big\langle E_-E_+(X^kY^{n-k}), X^kY^{n-k}\big\rangle \\ (n-k)^2\|X^{k+1}Y^{n-k-1}\|^2 = (k+1)(n-k)\|X^kY^{n-k}\|^2\end{gathered}$$

Iterating:

$$\begin{aligned}\|X^kY^{n-k}\|^2 &= \frac{n-k}{k+1}\cdot\|X^{k+1}Y^{n-k-1}\|^2 = \cdots = \frac{(n-k)(n-k-1)\ldots 1}{(k+1)(k+2)\ldots n}\cdot\|X^n\|^2 \\ &= \frac{(n-k)!k!}{n!}\end{aligned}$$

Conclusion:

$$e_{n,\,k} := \left(\begin{array}{c} n \\ k \end{array}\right)^{1/2} X^kY^{n-k}, \quad 0 \leq k \leq n$$

is an orthonormal basis for $\mathcal{H}_n$ with respect to the $SU(2)$-invariant inner product. Keep in mind that $e_{n,k}$ is of weight $2k-n$.

4. Decomposition of $L^2(SU(2))$

4.1. **Orthonormal Basis.** As a consequence of Peter-Weyl theorem, we know that the following functions

$$\Psi_{n,j,k}(g) = (n+1)^{1/2} < e_{nj}, \sigma_n(g) e_{nk} >, \quad n \geq 0, 0 \leq j, k \leq n$$

form an orthonormal Hilbert basis for $L^2(SU(2), d\mu_{Haar})$.

4.2. **Explicit formula.** Let $g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2)$, with $|a|^2 + |b|^2 = 1$. Then

$$\begin{aligned}
\sigma_n(g) e_{nk} &= \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} (aX - \bar{b}Y)^k (bX + \bar{a}Y)^{n-k} \\
&= \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} \left\{ \sum_{t=0}^{k} \begin{pmatrix} k \\ t \end{pmatrix} a^t (-\bar{b})^{k-t} X^t Y^{k-t} \right\} \cdot \left\{ \sum_{s=0}^{n-k} \begin{pmatrix} n-k \\ s \end{pmatrix} \bar{a}^{n-k-s} b^s X^s Y^{n-k-s} \right\} \\
&= \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} \underline{X^j Y^{n-j}} \sum_{\underline{s+t=j}} \begin{pmatrix} k \\ t \end{pmatrix} \cdot \begin{pmatrix} n-k \\ s \end{pmatrix} a^t (-\bar{b})^{k-t} b^s \bar{a}^{n-k-s} + \ldots
\end{aligned}$$

Therefore

$$\Psi_{n,j,k} = \sqrt{n+1} \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} \begin{pmatrix} n \\ j \end{pmatrix}^{-1/2} \sum_{s+t=j} \begin{pmatrix} k \\ t \end{pmatrix} \begin{pmatrix} n-k \\ s \end{pmatrix} a^t (-\bar{b})^{k-t} b^s \bar{a}^{n-k-s}$$

4.3. **Parametrization.** Using the parametrization

$$a = (\cos\phi)\, e^{i\theta_1}, \quad b = (\sin\phi)\, e^{i\theta_2}, \quad \theta_1, \theta_2, \phi \in [0, 2\pi)$$

we have

$$\begin{aligned}
\Psi_{n,j,k}(\theta_1, \theta_2, \phi) = \sqrt{n+1} & \begin{pmatrix} n \\ k \end{pmatrix}^{1/2} \begin{pmatrix} n \\ j \end{pmatrix}^{-1/2} e^{i(j+k-n)\theta_1} e^{i(j-k)\theta_2} \\
& \times \sum_{s+t=j} \begin{pmatrix} k \\ t \end{pmatrix} \begin{pmatrix} n-k \\ s \end{pmatrix} (\cos\phi)^{n-k-s+t} (\sin\phi)^{k-t+s} \qquad [0 \leq t \leq k, 0 \leq s \leq n-k]
\end{aligned}$$

5. Eigenfunctions of the Laplace-Beltrami operator

5.1. **Casimir Operator.** Consider the following element

$$\Omega = H^2 - 2H + 4E_+E_- = H^2 + 2H + 4E_-E_+$$

It is not hard to check that Ω commutes with H, E_+, E_-, hence

$$\sigma_n(\Omega) := \sigma_n(H)^2 - 2\sigma_n(H) + 4\sigma_n(E_+)\sigma_n(E_-) : \mathcal{H}_n \to \mathcal{H}_n$$

is an intertwining operator. By Schur's lemma there exists a constant $c(n)$ such that

$$\sigma_n(\Omega) v = c(n) v, \quad \forall v \in \mathcal{H}_n$$

By evaluating $\sigma_n(\Omega)$ on the highest weight vector we obtain $c(n) = n^2 + 2n$.

5.2. **Invariant differential operator.** Since $R_*(\Omega)\Psi_{n,j,k}(g) = (n+1)^{1/2} < e_{nj}, \sigma_n(g)\sigma_n(\Omega)e_{nk} >$ we have

$$R_*(\Omega)\Psi_{n,j,k}(g) = (n^2 + 2n)\Psi_{n,j,k}$$

We compute the operator $R_*(\Omega)$ and we remark in advance that coincides with the standard Laplace-Beltrami operator on $\mathbb{S}^3$.

5.3. **Computation of $R_*(\Omega)$.** Using the above parametrization, we regard a function f on $SU(2)$ as a function $f(\theta_1, \theta_2, \phi)$. We compute separately $R_*(H)$, $R_*(E_+)$ and $R_*(E_-)$ as differential operators in θ_1, θ_2, ϕ.

5.3.1. For $\begin{bmatrix} i & \\ & -i \end{bmatrix} \in su(2)$ we have

$$\begin{aligned} R_*\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} f(\theta_1,\theta_2,\phi) &= \frac{d}{dt}R(\exp t\begin{bmatrix} i & \\ & -i \end{bmatrix}) f(\theta_1,\theta_2,\phi)|_{t=0} \\ &= \frac{d}{dt}R\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} f(\theta_1,\theta_2,\phi)|_{t=0} \end{aligned}$$

Since

$$\begin{aligned} (\theta_1,\theta_2,\phi)\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix} &= \begin{bmatrix} (\cos\phi)e^{i\theta_1} & (\sin\phi)e^{i\theta_2} \\ -(\sin\phi)e^{-i\theta_1} & (\cos\phi)e^{-i\theta_1} \end{bmatrix} \cdot \begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} (\cos\phi)e^{i(\theta_1+t)} & (\sin\phi)e^{i(\theta_2-t)} \\ * & * \end{bmatrix} = (\theta_1+t,\theta_2-t,\phi) \end{aligned}$$

This means

$$R_*\begin{bmatrix} i & \\ & -i \end{bmatrix} f(\theta_1,\theta_2,\phi) = \frac{d}{dt}f(\theta_1+t,\theta_2-t,\phi)|_{t=0} = (\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2})f(\theta_1,\theta_2,\phi)$$

Therefore

$$R_*(H) = \frac{1}{i}R_*\begin{bmatrix} i & \\ & -i \end{bmatrix} = \frac{1}{i}(\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2})$$

5.3.2. Similarly

$$\begin{aligned} R_*\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} &= \tan\phi\sin(\theta_1-\theta_2)\frac{\partial}{\partial\theta_1} + \cot\phi\sin(\theta_1-\theta_2)\frac{\partial}{\partial\theta_2} + \cos(\theta_1-\theta_2)\frac{\partial}{\partial\phi} \\ R_*\begin{bmatrix} & i \\ i & \end{bmatrix} &= \tan\phi\cos(\theta_1-\theta_2)\frac{\partial}{\partial\theta_1} + \cot\phi\cos(\theta_1-\theta_2)\frac{\partial}{\partial\theta_2} - \sin(\theta_1-\theta_2)\frac{\partial}{\partial\phi} \end{aligned}$$

Therefore

$$\begin{aligned} R_*(E_+) &= \frac{1}{2}\left\{R_*\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} - iR_*\begin{bmatrix} & i \\ i & \end{bmatrix}\right\} = \frac{-ie^{i(\theta_1-\theta_2)}}{2}\left\{\tan\phi\frac{\partial}{\partial\theta_1} + \cot\phi\frac{\partial}{\partial\theta_2} + i\frac{\partial}{\partial\phi}\right\} \\ R_*(E_-) &= \frac{1}{2}\left\{-R_*\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} - iR_*\begin{bmatrix} & i \\ i & \end{bmatrix}\right\} = \frac{-ie^{i(\theta_2-\theta_1)}}{2}\left\{\tan\phi\frac{\partial}{\partial\theta_1} + \cot\phi\frac{\partial}{\partial\theta_2} - i\frac{\partial}{\partial\phi}\right\} \end{aligned}$$

5.3.3. *Conclusion.*

$$R_*(\Omega) = -\frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\theta_1^2} - \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta_2^2} - 2\cot(2\phi)\frac{\partial}{\partial\phi} - \frac{\partial^2}{\partial\phi^2}$$

5.4. **Laplace-Beltrami Operator.** The Riemannian metric induced from $\mathbb{S}^3 \subset \mathbb{R}^4$ is $ds^2 = \cos^2\phi d\theta_1^2 + \sin^2\phi d\theta_2^2 + d\phi^2$. A straight-forward computation shows that the Laplace-Beltrami operator is

$$\begin{aligned} \Delta_{\mathbb{S}^3} &= \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G}g^{ij}\frac{\partial}{\partial x_j}\right) = \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\theta_1^2} + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta_2^2} + 2\cot(2\phi)\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} \\ &= -R_*(\Omega) \end{aligned}$$

We now see that the functions $\Psi_{n,j,k}$ are eigenvalues of the Laplace-Beltrami operator:

$$-\Delta_{\mathbb{S}^3}\Psi_{n,j,k} = (n^2+n)\Psi_{n,j,k}, \quad n \geq 0, \quad 0 \leq j,k, \leq n$$

In particular each eigenvalue $n(n+2)$ has multiplicity $(n+1)^2$.