11. SPECIAL FUNCTIONS ON THE SPHERE

1. Relation between $SU(2)$ and $SO(3)$

1.1. **Killing form.** The Lie algebra $su(2)$ has an $SU(2)$-invariant positive-definite pairing given by

$$B(X,Y) = -\frac{1}{2}\,\mathrm{Tr}(XY)$$

where on the right-hand side we have matrix multiplication. It is easy to check that $B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X,Y)$, while definiteness can be seen from

$$B(X,X) = -\frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} x_{jk}x_{kj} = \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} |x_{jk}|^2, \quad X = (x_{jk}) \in su(2)$$

In particular $e_1 = \begin{bmatrix} & i \\ i & \end{bmatrix}$, $e_2 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$, $e_3 = \begin{bmatrix} i & \\ & -i \end{bmatrix}$ is an orthonormal basis (over $\mathbb{R}$) for $su(2)$.

1.2. **The adjoint map.** The choice of the basis $\{e_1, e_2, e_3\}$ identifies $su(2) \simeq \mathbb{R}^3$ and hence

$$\mathrm{Ad} : SU(2) \to GL(su(2)) \simeq GL(3,\mathbb{R})$$

Since the Killing form is invariant under $SU(2)$, we have a map

$$\phi : SU(2) \to SO(3)$$

Clearly, $\ker(\phi) = \{g \in SU(2) : \mathrm{Ad}(g) = I\} = Z(SU(2)) = \{\pm I_2\}$.

1.3. ϕ **is onto.** One way to argue this is as follows: since $\ker(\phi_*) = Lie(\ker(\phi)) = 0$, $\phi_* : su(2) \to so(3)$ is an isomorphism, hence ϕ is a local diffeomorphism. But this implies that $\phi(SU(2))$ is an open subset of $SO(3)$ and hence closed as well. Since $SO(3)$ is connected, we have $\phi(SU(2)) = SO(3)$. Therefore $SO(3) = SU(2)/\{\pm I_2\}$.

1.4. **Explicit formula.** To determine ϕ explicitly, we have to determine $\mathrm{Ad}(g)$ as a 3×3 matrix in the $e_1, e_2, e_3\}$-basis. A straightforward computation yields

$$\phi\left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\right) = \begin{bmatrix} Re(a^2-b^2) & Im(a^2+b^2) & -Re(2ab) \\ -Im(a^2-b^2) & Re(a^2+b^2) & Im(2ab) \\ a\bar{b}+\bar{a}b & \frac{1}{i}(a\bar{b}-\bar{a}b) & |a|^2-|b|^2 \end{bmatrix}$$

2. The dual of $SO(3)$

Let $(\pi, V) \in \widehat{SO(3)}$. We have

$$\begin{array}{ccc} SU(2) & & \\ \downarrow{\scriptstyle \phi} & & \\ SO(3) & \xrightarrow{\ \pi\ } & GL(V) \end{array}$$

Since ϕ is surjective, $\pi \circ \phi$ is an irreducible representation of $SU(2)$. This means the inclusion $\widehat{SO(3)} \subset \widehat{SU(2)} = \{\sigma_n : n \geq 0\}$. Conversely, an element $\sigma_n \in \widehat{SU(2)}$ comes from an irreducible representation of $SO(3)$ if and only if $\sigma_n(\pm I_2) = I$. Since $\sigma_n(-I_2) = \mathrm{diag}(e^{in\pi}, e^{i(n-2)\pi}, \ldots, e^{-in\pi})$, this happens when $n \equiv (\mathrm{mod}\, 2)$. Therefore each σ_n, $n = 0, 2, 4$ etc. corresponds to an irreducible representation of $SO(3)$. The explicit action of $SO(3)$ on $\mathcal{H}_n$ is given by $\sigma_n \circ \phi^{-1}$.

3. Relation between $SU(2)$ and $\mathbb{S}^2$

The standard action of $SO(3)$ on $\mathbb{R}^3$ preserves the unit sphere $\mathbb{S}^2$, hence we have an action of $SO(3)$ on $\mathbb{S}^2$. This action is transitive. Since $\phi : SU(2) \to SO(3)$ is surjective, we have a transitive action of $SU(2)$ on $\mathbb{S}^2$ by $(g, P) \mapsto \phi(g)P$. If we fix $N = (0, 0, 1)$ as reference point we obtain the surjective map $p : SU(2) \to \mathbb{S}^2$,

$$p(g) = \phi(g)N = (Re(2ab), Im(2ab), |a|^2 - |b^2|)$$

The stabilizer of N in $SU(2)$ is the diagonal torus $T = \begin{bmatrix} * & \\ & * \end{bmatrix} \subset SU(2)$, hence we have the identification

$$p : SU(2)/T \simeq \mathbb{S}^2$$

4. Special functions on $\mathbb{S}^2$

In this section we will use the coset-space identification $\mathbb{S}^2 = SU(2)/T$ to determine an orthonormal-basis for $L^2(\mathbb{S}^2)$. Next, we will identify this functions with the spherical harmonics and then prove that they are eigenfunctions of the Laplace operator.

4.1. **A consequence of Peter-Weyl.** First, we identify functions on $\mathbb{S}^2$ with functions on $SU(2)$ which are T-right invariant. Namely, if $F : SU(2) \to \mathbb{C}$ is such that $F(gt) = F(g)$, $\forall g \in SU(2), t \in T$, then the function $f : \mathbb{S}^2 \to \mathbb{C}$ defined by $f(P) = F(g)$ if $P = p(g)$ is well defined. It is straightforward to determine that the correspondence $F \leftrightarrow f$ is bijective. It is less obvious that $\|F\|_{SU(2)} = c\|f\|_{\mathbb{S}^2}$, where $c = 2\sqrt{\pi}$ is an absolute constant [homework].

Therefore

$$L^2(\mathbb{S}^2) = L^2(SU(2))^{r(T)} = \left(\widehat{\oplus}_{n\geq 0}\mathcal{H}_n^* \otimes \mathcal{H}_n\right)^{r(T)} = \widehat{\oplus}_{n\geq 0}\mathcal{H}_n^* \otimes \mathcal{H}_n^T$$

Since $\mathcal{H}_n^T = \begin{cases} \mathbb{C} \cdot e_{n,n/2}, & n \equiv 0 (\mathrm{mod}\, 2) \\ 0, & \equiv 1 (\mathrm{mod}\, 2) \end{cases}$, we see that $L^2(\mathbb{S}^2) \simeq \widehat{\oplus}_{2|n}\mathcal{H}_n^*$. In particular the functions $\Psi_{n,j,n/2}$ descend to functions $Y_{n,j}$ on $\mathbb{S}^2$ by

$$Y_{n,j}(P) = \Psi_{n,j,n/2}(g), \quad \text{if } p(g) = P$$

and the set $Y_{n,j}$: n positive and even, $0 \leq j \leq n$ form an orthonormal basis for $L^2(\mathbb{S}^2)$ (up to a scalar factor).

4.2. **Explicit formula.** Let $P = (x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\theta) \in \mathbb{S}^2$. Let $g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ such that $p(g) = P$. Then $x + iy = 2ab$ and $|a|^2 - |b|^2 = z$. Let $n = 2m \geq 0$, with $m \in \mathbb{Z}$ and $0 \leq j \leq 2m$. Recall:

$$\Psi_{2m,j,m} = \sqrt{2m+1} \begin{pmatrix} 2m \\ m \end{pmatrix}^{1/2} \begin{pmatrix} 2m \\ j \end{pmatrix}^{-1/2} \sum_{s+t=j} \begin{pmatrix} m \\ t \end{pmatrix} \begin{pmatrix} m \\ s \end{pmatrix} a^t(-\bar{b})^{m-t}b^s\bar{a}^{m-s} \quad [0 \leq s, t \leq m]$$

Since: $a^t\bar{a}^{m-s}b^s\bar{b}^{m-t} = |a|^{-2s}|b|^{2s-2j}(ab)^j(\bar{a}\bar{b})^m = (\frac{1+z}{2})^{-s}(\frac{1-z}{2})^{s-j}(\frac{1}{2}\sin\phi\, e^{i\theta})^j(\frac{1}{2}\sin\phi\, e^{-i\theta})^m$

$$= 2^{-m}e^{i(j-m)\theta}(1+z)^{-s}(1-z)^{s-j}(\sin\phi)^{j+m}, \quad z = \cos\phi$$

$$2^{-m}e^{i(j-m)\theta}(1-z)^{s-\frac{j-m}{2}}(1+z)^{t-\frac{j-m}{2}}$$

the corresponding function on the sphere is

$$Y_{2m,j}(\theta, \phi) = \frac{(-1)^{(j-m)/2}}{2^m m!}\sqrt{j!(2m-j)!(2m+1)}\, e^{i(j-m)\theta} \sum_{s+t=j} \begin{pmatrix} m \\ t \end{pmatrix} \begin{pmatrix} m \\ s \end{pmatrix} (z-1)^{s-\frac{j-m}{2}}(z+1)^{t-\frac{j-m}{2}}$$

4.3. **Spherical Harmonics.** The standard spherical harmonics on the sphere are given by

4.3.1. *Definition.*

$$Y_l^k(\theta, \phi) = \frac{1}{\sqrt{2\pi}}e^{ik\theta}P_l^k(\cos\phi), \quad P_l^k(z) = \sqrt{\frac{(l+k)!}{(l-k)!}}\sqrt{l+1/2}\,\frac{1}{2^l l!}(1-z^2)^{-k/2}\frac{d^{l-k}}{dz^{l-k}}(z^2-1)^l$$

4.3.2. *Lemma.*

$$\frac{d^k}{dz^k}\left[(z^2-1)^N\right] = k! \sum_{a+b=2N-k} \binom{N}{k}\binom{N}{b}(z-1)^a(z+1)^b \quad [0 \le a, b \le N]$$

The proof of the lemma is elementary. Using the lemma we find

$$Y_l^k(\theta,\phi) = \frac{e^{ik\theta}}{2\sqrt{\pi}}\frac{(-1)^{k/2}}{2^l l!}\sqrt{(l+k)!(l-k)!(2l+1)} \sum_{a+b=l+k}\binom{l}{a}\binom{l}{b}(z-1)^{a-k/2}(z+1)^{b-k/2}$$

By comparing the formulas for $Y_{2m,j}$ and Y_k^l we see that

$$Y_{2m,j} = 2\sqrt{\pi}\, Y_m^{j-m}, \quad Y_l^k = \frac{1}{2\sqrt{\pi}}\, Y_{2l,l+k}$$

4.4. **Differential equations satisfied by the spherical harmonics.** Since $\Psi_{2m,j,m}$ is a matrix coefficient function from σ_{2m}, we know that

$$L_*(\Omega)\Psi_{2m,j,m} = (2m)^2 + 2\cdot 2m = 4m(m+1)$$

We will determine an explicit formula for the left-action of $L_*(\Omega)$ on (functions on) $SU(2)/T = \mathbb{S}^2$ in (θ,ϕ)-coordinates.

Assume f is a function on $SU(2)/T$. Then $f\left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}\right) = f(\theta,\phi)$, where $2ab = \sin\phi e^{i\theta}$ and $|a|^2 - |b|^2 = \cos\phi$. Then

$$\begin{aligned} L_*(H)f = \frac{1}{i}L_*(iH)f &= \frac{d}{idt} f\left(\begin{bmatrix} e^{-it} & \\ & e^{it}\end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}\right)\Big|_{t=0} = \frac{d}{idt} f(\theta - 2t,\phi)|_{t=0} \\ &= 2i\frac{\partial}{\partial\theta}f \end{aligned}$$

Carrying out similar computations for $L_*(E_\pm)$ we obtain

$$-\frac{1}{4}L_*(\Omega) = \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\phi^2} + (\cot\phi)\frac{\partial}{\partial\phi} = \Delta_{\mathbb{S}^2}$$

the Laplace operator on the sphere. Therefore the functions $Y_{2m,j}$ satisfy the differential equation

$$-\Delta_{\mathbb{S}^2} Y_{2m,j} = m(m+1)Y_{2m,j}$$

which corresponds to the standard $(\Delta_{\mathbb{S}^2} + l(l+1))Y_l^k = 0$.