12. THE DUAL OF $SU(3)$

1. Preliminaries

1.1. **The action of the torus.** Here G = compact, connected Lie group. T = fixed maximal torus. Assume (π, V) is a representation of G. Then under the action of T we have the decomposition

$$V = \bigoplus_{\phi \in T^*} V(\phi)$$

where for T^* is the set of one-dimensional characters and $V(\phi) = \{v \in V : \pi(t)v = \phi(t)v, \ \forall\, t \in T\}$.

1.2. **The unit lattice.** Recall: $\exp : \mathfrak{t} \to T$ is an onto homomorphism (of abelian groups), hence $T \simeq \mathfrak{t}/L$, where $L = \{H \in \mathfrak{t} : \exp(H) = e\}$. Since T is compact, L is a lattice in $\mathfrak{t}$. We call it the *unit lattice.*

1.3. **The structure of T^*.** Let $\phi : T \to \mathbb{C}^\times$ a (continuous) group homomorphism and $\phi_* : \mathfrak{t} \to \mathbb{C}$ the associated homomorphism of Lie algebras. Since $\mathfrak{t}$ is abelian, ϕ_* is simply an $\mathbb{R}$-linear map. The following diagram is commutative

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\phi_*} & \mathbb{C} \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ T & \xrightarrow{\phi} & \mathbb{C}^\times \end{array}$$

and the only restriction on ϕ_* is that $\exp(\phi_*(L)) = 1$. In other words, $\phi_* \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{C})$ and $\phi_*(L) \in 2\pi i\mathbb{Z}$. We switch to the notation

$$\lambda := \phi_*, \quad e^\lambda := \phi \in T^*$$

Since $\lambda(L) \subset 2\pi i\mathbb{Z}$ and L spans $\mathfrak{t}$ (over $\mathbb{R}$), it follows that $\lambda(\mathfrak{t}) \subset i\mathbb{R}$. In other words the set of such λ is

$$\Lambda := \{\lambda \in i\mathfrak{t}^* : \lambda(L) \subset 2\pi i\mathbb{Z}\}, \quad \text{where} \quad \mathfrak{t}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$$

Since L is a lattice in $\mathfrak{t}$, Λ is a lattice in $i\mathfrak{t}^*$ (dual lattice). It is called *the weight lattice.* We have now established the bijection

$$\Lambda \longleftrightarrow T^*, \quad \lambda \leftrightarrow e^\lambda, \quad e^\lambda(\exp H) := e^{\lambda(H)}, \forall\, H \in \mathfrak{t}$$

1.4. **Weights.** If (π, V) is a representation, it is straightforward to check that

$$V(e^\lambda) = \{v \in V : Hv = \lambda(H)v, \quad \forall H \in \mathfrak{t}\}$$

We will use the notation $V(\lambda)$ instead of $V(e^\lambda)$ and re-write the decomposition under the action of the torus as

$$V = \oplus_{\lambda \in \mathcal{E}(\pi)} V(\lambda)$$

where $\mathcal{E}(\pi) := \{\lambda : V(\lambda) \neq 0\} \subset \Lambda$ is the set of (non-trivial) weights.

1.5. **Roots.** If we decompose $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ under the *adjoint action* of T we obtain a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha$$

where $\Phi \subset \Lambda - \{0\}$ is the set of non-zero weights of the adjoint representation and

$$\mathfrak{g}^\alpha := \mathfrak{g}_{\mathbb{C}}(\alpha) = \{X \in \mathfrak{g}_{\mathbb{C}} : [H, X] = \alpha(H)X, \quad \forall H \in \mathfrak{t}\}$$

Note that the weight zero subspace is

$$\mathfrak{g}_{\mathbb{C}}(0) := \{X \in \mathfrak{g}_{\mathbb{C}} : [H, X] = 0, \quad \forall H \in \mathfrak{t}\} = \mathfrak{t}_{\mathbb{C}}$$

since $\mathfrak{t}_{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}_{\mathbb{C}}$. Therefore $\mathcal{E}(\mathrm{Ad}) = \{0\} \cup \Phi$.

1.6. **Action of Φ on the weights.** The analogue of the raising and lowering operators $E_\pm$ is the following fact:

$$\text{for} \quad X_\alpha \in \mathfrak{g}^\alpha : \quad \pi_*(X_\alpha) : V(\lambda) \to V(\lambda + \alpha)$$

Proof: let $H \in \mathfrak{t}$ arbitrary. Then

$$H(X_\alpha v) = X_\alpha(HV) + [H, X_\alpha]v = \lambda(H)X_\alpha v + \alpha(H)X_\alpha v = (\lambda + \alpha)(H)X_\alpha v$$

2. The structure of $SU(3)$

2.1. **Lie algebra.** $\mathfrak{g} = su(3) = \{A \in gl(3,\mathbb{C}) : A^* + A = 0\}$, where $A^* = \overline{A}^t$. Since every trace-less matrix $X \in sl(3,\mathbb{C})$ can be written as

$$X = \frac{1}{2}(X - X^*) + \frac{i}{2i}(X - X^*) = A + iB$$

with $A, B \in su(3)$, it means that the complexified Lie algebra is

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = sl(3,\mathbb{C})$$

2.2. **Maximal torus.** In the case of $G = SU(3)$ on can choose the maximal torus

$$T := \Big\{ \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) : \theta_j \in \mathbb{R}, \sum \theta_j = 0 \Big\}, \quad \text{where} \quad \operatorname{diag}(x_1,x_2,x_3) := \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}$$

Clearly $\mathfrak{t} = \{\operatorname{diag}(i\theta_1, i\theta_2, i\theta_3) : \theta_j \in \mathbb{R}, \sum \theta_j = 0\}$, and $\mathfrak{t}_{\mathbb{C}} = \{\operatorname{diag}(w_1,w_2,w_3) : w_j \in \mathbb{C}, \sum w_j = 0\}$.
In particular $i\mathfrak{t} = \{\operatorname{diag}(x_1,x_2,x_3) : x_j \in \mathbb{R}, \sum x_j = 0\}$.

With respect to the Killing form $B(X,Y) = \frac{1}{2}\operatorname{Tr}(XY)$, we can choose for $i\mathfrak{t}$ the orthonormal basis $\{h_1,h_2\}$, with $h_1 = \operatorname{diag}(1,-1,0)$ and $h_2 = \operatorname{diag}(\sqrt{3}/3, \sqrt{3}/3, -2\sqrt{3}/3)$.

2.3. **Unit lattice.** The unit lattice is given by

$$L = \{\operatorname{diag}(2\pi i n_1, 2\pi i n_2, 2\pi i n_3) : n_j \in \mathbb{Z}, \sum n_j = 0\}$$

Since $\operatorname{diag}(n_1,n_2,n_3) = n_1 \operatorname{diag}(1,-1,0) + (n_1+n_2)\operatorname{diag}(0,1,-1)$, $\frac{1}{2\pi i}L$ is generated (over $\mathbb{Z}$) by

$$H_{12} := \operatorname{diag}(1,-1,-0) = h_1, \quad H_{23} := \operatorname{diag}(0,1,-1) = \frac{1}{2}h_1 + \frac{\sqrt{3}}{2}h_2$$

In the $\{h_1,h_2\}$-plane the lattice $\frac{1}{2\pi i}L \subset i\mathfrak{t}$ is generated by $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. This is a hexagonal lattice containing the roots of order 6.

2.4. **The linear dual.** Let $\{f_1,f_2\}$ the basis for $i\mathfrak{t}^*$ dual to $\{h_1,h_2\}$: $f_j(h_k) = \delta_{jk}$, for $j,k \in \{1,2\}$. The inner product on $i\mathfrak{t}^* := \operatorname{Hom}_{\mathbb{R}}(i\mathfrak{t},\mathbb{R})$ is such that the dual basis $\{f_1,f_2\}$ is orthonormal. Thus for $\lambda,\mu \in i\mathfrak{t}^*$ we have $\lambda = \lambda(h_1)f_1 + \lambda(h_2)f_2$ and the inner product is $(\lambda,\mu) = \lambda(h_1)\mu(h_1) + \lambda(h_2)\mu(h_2)$.

Consider the linear functionals $l_1,l_2,l_3 \in i\mathfrak{t}^*$ given by $l_j(\operatorname{diag}(x_1,x_2,x_3)) = x_j$, $1 \le j \le 3$. Clearly these span $i\mathfrak{t}^*$ and $l_1+l_2+l_3 = 0$. The relation between l_j and f_j is

$$l_1 = f_1 + \frac{\sqrt{3}}{2}f_3, \quad l_2 = -f_1 + \frac{\sqrt{3}}{2}f_2, \quad l_3 = -\frac{2\sqrt{3}}{2}f_2$$

When λ and μ are given as linear combinations of l_j, say $\lambda = \sum_{i=1}^3 \lambda_i l_i$ and $\mu = \sum_{j=1}^3 \mu_j l_j$ with $\lambda_j, \mu_j \in \mathbb{R}$ then $\lambda(f_1) = \lambda_1 - \lambda_2$, $\lambda(f_2) = \frac{\sqrt{3}}{3}(\lambda_1 + \lambda_2 - 2\lambda_3)$ and it is a straightforward exercise to check that

$$(\lambda,\mu) = 2\sum_{j=1}^{3}\lambda_j\mu_j - \frac{2}{3}(\sum_{j=1}^{3}\lambda_j)(\sum_{j=1}^{3}\mu_j)$$

2.5. **Weight lattice.** Since $\frac{1}{2\pi i}L = \mathbb{Z}H_{12} \oplus \mathbb{Z}H_{23}$, the weight lattice is given by

$$\Lambda = \{\lambda \in i\mathfrak{t}^* : \lambda(H_{12}) \in \mathbb{Z}, \quad \lambda(H_{23}) \in \mathbb{Z}\}$$

The l_j take values 0 and 1 on H_{ij}, hence it is immediate that Λ is spanned by l_1,l_2,l_3. That is, $\Lambda = \mathbb{Z}l_1 \oplus \mathbb{Z}l_3 = \mathbb{Z}l_1 \oplus \mathbb{Z}l_2$.

2.6. **Roots.** Let E_{ij} be the matrix whose (i,j)-entry is 1 and all other entries are 0. Since $[H,E_{ij}] = (x_i - x_j)E_{ij}$ for every $H = \operatorname{diag}(x_1,x_2,x_3) \in i\mathfrak{t}$, it means that E_{ij} is a root vector corresponding to the root $\alpha_{ij} := l_i - l_j$. Writing $\mathfrak{g}^{ij}$ instead of $\mathfrak{g}^{\alpha_{ij}}$ we have the the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{i \ne j} \mathfrak{g}^{ij}, \quad \mathfrak{g}^{ij} = \mathbb{C}E_{ij}$$

where $\Phi = \{\alpha_{ij} : i \ne j\}$, $\alpha_{ij} := l_i - l_j$.

3. The highest weight

3.1. **Positive half-plane.** Let δ an arbitrary line in $i\mathfrak{t}^*$ which passes through the origin but doesn't contain any other point of the lattice Λ. (Such a line exists by arranging for its slope to fall outside a countable set.) The line δ separates the plane $i\mathfrak{t}^*$ into two half-planes. We choose one such half-plane and call it the positive half-plane. Then Φ is the disjoint union $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ is the set of roots in the positive half-plane, and $\Phi^- = -\Phi^+$. We may assume without loss of generality that the line δ is such that $\Phi^+ = \{\alpha_{ij} : 1 \leq i < j \leq 3\}$.

3.2. **Definition.** Assume now $(\pi, V) \in \widehat{G}$ is an irreducible representation of G. Let $\lambda \in \mathcal{E}(\pi)$ be a weight of π satisfying the following two properties

- λ is in the positive half-plane (the one containing Φ^+)
- λ maximizes the distance to the line δ

3.3. **Uniqueness.** We prove that a weight with the above two properties is necessarily unique. For assume that there exists two distinct λ, λ' in the positive half-plane, of equal distance to the line δ. This implies that δ passes through the non-zero weight $\lambda - \lambda' \in \Lambda$, contradiction. Hence λ with the said properties is unique, and we call it *the highest weight of* π. Note that λ depends only on the choice of a positive half-plane.

Goal: in what follows we will aim to express all the fundamental features of π (dimension, character) in terms of its highest weight.

3.4. **Further Properties.**

3.5. We know that $E_{12} : V(\lambda) \to V(\lambda + \alpha_{12})$. However, λ being the highest weight it prevents $\lambda + \alpha_{12}$ to be a (non-trivial) weight of π, since $\lambda + \alpha_{12}$ is in the positive half-plane and is farther away from the dividing line δ than λ is. Therefore $V(\lambda+\alpha_{12}) = 0$. Similarly $V(\lambda+\alpha_{23}) = 0$ as well. This simply means that

$$E_{12}v = E_{23}v = 0 \quad \text{for} \quad v \in V(\lambda)$$

3.6. Since π is irreducible, this implies that V is an irreducible $\mathfrak{g}_{\mathbb{C}}$-module. In particular, this says that all the elements of V can be obtained by successive applications of $X \in \mathfrak{g}_{\mathbb{C}}$ on a fixed vector $v_0 \in V(\lambda)$. Hence for every $\mu \in \mathcal{E}(\pi)$, the weight space $V(\mu)$ is spanned by images of various

$$X_1 X_2 \dots X_k : V(\lambda) \to V(\mu)$$

Since E_{ij}, with $i < j$ annihilate $V(\lambda)$, we can take the X_k's above to be scalar multiples of $H \in \mathfrak{t}$ and E_{ji} with $i < j$ (negative roots). [1] By hitting $V(\lambda)$ with negative roots we only obtain weights of the type

$$\mu = \lambda - m\alpha_{12} - n\alpha_{23}, \quad m, n \in \mathbb{Z}_{\geq 0}$$

Also, it is not hard to see that the irreducibility of V implies that $\dim V(\lambda) = 1$.

Observation 1. $\mathcal{E}(\pi)$ is confined to a cone with the vertex at λ and sides parallel to $-\alpha_{12}$ and $-\alpha_{23}$.

3.7. The next step is to investigate possible roots of the form $\lambda - k\alpha_{12}$ and $\lambda - k\alpha_{23}$ lying on the boundary of the cone at λ. Let $p \geq 0$ be the greatest positive integer such that $\lambda - p\alpha_{12}$ is a weight.

3.8. The subspace $V_{12} := \oplus_{0 \leq k \leq p} V(\lambda - k\alpha_{12})$ is invariant under action of H_{12}, E_{12}, E_{21}. Since these form and $sl(2)$-triple:

$$[H_{12}, E_{12}] = 2E_{12}, \quad [H, E_{21}] = -2E_{21}, \quad [E_{12}, E_{21}] = H_{12}$$

we have that V_{12} is an $sl(2)$-module of highest weight $m = \lambda(H_{12})$ and lowest weight $m' = (\lambda - p\alpha_{12})(H_{12})$. Necessarily $m' = -m \Rightarrow \; p = \lambda(H_{12})$ and hence

Observation 2. $p = \lambda(H_{12}) \in \mathbb{Z}_{\geq 0}$ is the largest positive integer such that $\lambda, \lambda - \alpha_{12}, \dots, \lambda - p\alpha_{12} \in \mathcal{E}(\pi)$.

3.9. We note here that $\lambda(H_{12}) = \frac{2(\lambda, \alpha_{12})}{(\alpha_{12}, \alpha_{12})}$ so we can re-write

$$\lambda - p\alpha_{12} = \lambda - \frac{2(\lambda, \alpha_{12})}{(\alpha_{12}, \alpha_{12})}\alpha_{12} =: s_{12}\lambda$$

where $s_{12} : i\mathfrak{t}^* \to i\mathfrak{t}^*$ is the reflection across $\alpha_{12}^{\perp}$ (the line perpendicular on α_{12}) [homework].
Similarly, $s_{23}\lambda = \lambda - \lambda(H_{23}) \in \mathcal{E}(\pi)$.

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[1] This corresponds to 12.10, p. 167 of Fulton and Harris, *Representation Theory*.

3.10. So far we obtained that if $\lambda \in \mathcal{E}(\pi)$ is a highest weight, then $s_{12}\lambda$ and $s_{23}\lambda$ are also weights and $\mathcal{E}(\pi)$ is inside the cone with the vertex at λ and edges connecting λ to $s_{12}\lambda$ and $s_{23}\lambda$. But now we can change the choice of the positive half-plane, and apply the same procedure to $s_{12}\lambda$ and $s_{23}\lambda$ (now highest weights corresponding to the different choices of the positive cone) and keep reflecting... Since s_{12} and s_{23} generate a group W of order 6, we end up with a hexagon: the vertices of the hexagon are the W-orbit of λ, and $\mathcal{E}(\pi)$ is in the interior of the hexagon.

Definition. The group W generated by the reflections s_{ij}, $i \neq j$ in the $i\mathfrak{t}^*$ plane is called the Weyl group.

Observation 3. We summarize the previous paragraph:

$$\mathcal{E}(\pi) \subset \text{ConvexHull}(W \cdot \lambda) \ \cap \ (\lambda + \Phi \otimes \mathbb{Z}) \tag{1}$$

where $\Phi \otimes \mathbb{Z}$ is the lattice generated by Φ inside Λ, and $\lambda + \Phi \otimes \mathbb{Z}$ is the translation by the vector λ.

3.11. We fix a positive half-plane, such that $\Phi^+ = \{\alpha_{ij}, i < j\}$. Question: which vectors $\lambda \in \Lambda$ can occur as highest weights of some irreducible representation? The only conditions such a λ had to satisfy were

$$\lambda \in \Lambda \quad \text{and} \quad \lambda(H_{ij}) = \frac{2(\lambda, \alpha_{ij})}{(\alpha_{ij}, \alpha_{ij})} \in \mathbb{Z}_{\geq 0}, \quad \forall 1 \leq i < j \leq 3$$

Let $\lambda = \lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3$, with $\lambda_j \in \mathbb{R}$. Then $\lambda(H_{12}) = \lambda_1 - \lambda_2$ and $\lambda(H_{23}) = \lambda_2 - \lambda_3$ hence $\lambda_1 - \lambda_2 = m$ and $\lambda_2 - \lambda_3 = n$ are positive integers. We find that

$$\lambda = \lambda_1 l_1 - \lambda_2(l_1 + l_3) + \lambda_3 l_3 = m l_1 - n l_3, \quad m, n \in \mathbb{Z}_{\geq 0}$$

We denote by Λ^+ the set of such λ: it is the intersection of the lattice Λ with the cone spanned by l_1 and $-l_3$. So far we saw that to each $\pi \in \widehat{G}$ it corresponds a highest weight $\lambda \in \Lambda^+$. The converse is also true, and we have the following

3.12. **Theorem.** The correspondence $\widehat{G} \longleftrightarrow \Lambda^+$ which assigns to each irreducible representation G its highest weight, is a bijection.

Proof. One way to argue that the representation is uniquely determined by its weight is to prove that the weight determines its character. This is achieved (explicitly) by Weyl's character formula which will be discussed later. To prove that for each $\lambda \in \Lambda^+$ there exists a representation π_λ of highest weight λ, one can argue as in Fulton and Harris (p. 180-182): let π_{st} be the standard (irreducible) action of $SU(3)$ on $\mathbb{C}^3$. Then π_{st} is of highest weight l_1. The symmetric tensor product $\text{sym}^m \pi_{st}$ is irreducible of highest weight ml_1. Similarly, the dual representation π^*_{st} is of highest weight $-l_3$ and its symmetric tensor power $\text{sym}^n \pi^*_{st}$ is of highest weight $-nl_3$. Then $\text{sym}^m \pi_{st} \otimes \text{sym}^n \pi^*_{st}$ must contain an irreducible representation of highest weight $ml_1 - nl_3$.

Definition. In general, we denote by (π_λ, V_λ) a representation of highest weight λ. In the case $G = SU(3)$, $\sigma_{m,n}$ is the irreducible representation of highest weight $ml_1 - nl_3$, for $m, n \geq 0$, integers. We denote by $\chi_{m,n}$ its character.

4. Character Formula

4.1. The next question is that of determining the dimension of π_λ and the dimension of each of its weight spaces. The key to this is the character χ_λ of π_λ. For let $n_\mu = \dim V_\lambda(\mu)$. This is called the multiplicity of μ in π_λ. The multiplicities are encoded in the trace of π_λ since

$$\chi_\lambda|_T = \sum_{\mu \in \mathcal{E}(\pi)} n_\mu e^\mu$$

Hence determining χ_λ is key. Weyl's character formula achieves just that. We first state it in full generality, discuss an example in the case of $SU(3)$ and postpone its proof for the next lecture.

4.2. **Weyl character formula.** Let π_λ the irrep. of highest weight λ, and let χ_λ denote its trace. Since χ_λ is a class function and every elements of G is conjugated to an element of T, it means that χ_λ is completely determined by its restriction to T. The explicit formula for this restriction is

$$\chi_\lambda|_T = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\epsilon(w)$ is 1 if w is orientation preserving (product of an even number of reflections) and -1, otherwise. Note that this formula holds for an arbitrary compact group.

4.3. **Example.** We consider the case of $G = SU(3)$ and $\lambda = 2l_1 - l_3$. The irreducible representation in question is $\sigma_{2,1}$. For simplicity we will express everything in terms of l_1 and l_3:

$$\Phi^+ = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\} = \{2l_1 + l_3, -l_1 - 2l_3, l_1 - l_3\}, \quad \rho = \frac{1}{2}(\alpha_{12} + \alpha_{23} + \alpha_{13}) = \alpha_{13} = l_1 - l_3$$

Since $s_{12}(\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3) = \lambda_2 l_1 + \lambda_1 l_2 + \lambda_3 l_3$ and similarly for the other s_{ij} [exercise] we see that the Weyl group W can be identified with the permutation group S_3 and under this identification, $\epsilon(w)$ is the signature of the permutation:

$$w \in W \simeq S_3, \quad w(\sum_{j=1}^{3} \lambda_j l_j) = \sum_{j=1}^{3} \lambda_{w(j)} l_j, \quad \epsilon(w) = sgn(w)$$

Since in this particular example $\lambda + \rho = 3l_1 - 2l_3$, it means that the W-orbit of this can be determined as follows

- $w = e: \quad w(\lambda + \rho) = 3l_1 - 2l_3$
- $w = (12): \quad w(\lambda + \rho) = 3l_2 - 2l_3 = -3l_1 - 5l_3$
- $w = (23): \quad w(\lambda + \rho) = 3l_1 - 2l_2 = 5l_1 + 2l_3$
- $w = (13): \quad w(\lambda + \rho) = -2l_1 + 3l_3$
- $w = (123): \quad w(\lambda + \rho) = -2l_1 + 3l_2 = -5l_1 - 3l_3$
- $w = (132): \quad w(\lambda + \rho) = -2l_2 + 3l_3 = 2l_1 + 5l_3$

The denominator in Weyl's formula is

$$\Delta = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = e^{l_1 - l_3}(1 - e^{-2l_1 - l_3})(1 - e^{l_1 + 2l_3})(1 - e^{l_1 - l_3})$$

Therefore

$$\chi_{2,1} = \frac{e^{3l_1 - 2l_3} - e^{-3l_1 - 5l_3} - e^{5l_1 + 2l_3} - e^{-2l_1 + 3l_3} + e^{-5l_1 - 3l_3} + e^{2l_1 + 5l_3}}{e^{l_1 - l_3}(1 - e^{-2l_1 - l_3})(1 - e^{l_1 + 2l_3})(1 - e^{l_1 - l_3})}$$

Feeding this to Maple we obtain for the right-hand side

$$e^{3l_1 + l_3} + e^{2l_1 + 2l_3} + e^{2l_1 - l_3} + e^{l_1 + 3l_3} + e^{-2l_3} + e^{2l_3 - l_1} + e^{-2l_1} + e^{-2l_1 - 3l_3} + e^{-3l_1 - 2l_3} + 2e^{l_1} + 2e^{l_3} + 2e^{-l_3 - l_1}$$

This shows exactly what the weights are and their multiplicities. For example, l_1, l_3 and $l_2 = -l_1 - l_3$ are weights of multiplicity 2. Also, it tells us that $\dim \sigma_{2,1} = 15$.

4.4. **Note.** For $\lambda = 0$ we obtain a formula for the trace of the trivial representation

$$1 = \frac{\sum_{w \in W} \epsilon(w) e^{w\rho}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

This gives an identity for the denominator in Weyl formula

$$\Delta = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W} \epsilon(w) e^{w\rho}$$