12. WEYL'S CHARACTER FORMULA

1. Weyl's Integration formula

1.1. **Set-up.** G is a compact, connected Lie group, T a fixed maximal torus. The Haar measures dg on G and dt on T are normalized so that $\mathrm{vol}(G) = \int_G dg = 1$ and $\mathrm{vol}(T) = \int_T dt = 1$. Also, the coset space G/T is endowed with a left-invariant (under the G-action) measure $d\mu_{G/T}$ and we normalized it so that $\mathrm{vol}(G/T) = \int_{G/T} d\mu_{G/T} = 1$. Let $\Delta = e^{\rho}\prod_{\alpha\in\Phi^+}(1-e^{-\alpha}) = \sum_{w\in W} e^{w\rho}$ the denominator in Weyl's character formula. With the above normalizations we have the following

1.2. **Theorem.** Assume $f \in C(G)$ is a continuous function on G. Then

$$\int_G f(g)dg = \frac{1}{|W|}\int_{G/T}\int_T f(gtg^{-1})|\Delta(t)|^2 dt d\mu_{G/T}(gT)$$

We skip over the details of this proof. The formula is obtained by applying the change of variable formula to the map

$$\alpha : G/T \times T \to G, \quad \alpha(gT, h) = ghg^{-1}$$

The idea is that this function is surjective, since every element of G has a conjugate in T. Moreover (outside a negligible set) μ is a covering map whose fiber is $N_G(T)/T$. Part of the theory is to establish the bijection $N_G(T)/T \simeq W$. In the case of $SU(3)$ this is not hard to see since both can be identified with S_3. Hence the size of (almost all) fibers of α is $|W|$, and this explains the factor $\frac{1}{|W|}$ on the right-hand side. Then the factor $|\Delta|^2$ is simply the Jacobian of μ. [1]

1.3. **Corollary.** If $f \in C(G)_{class}$ is a class function, that is $f(xyx^{-1}) = f(x)$, $\forall x, y \in G$, then

$$\int_G f(g)dg = \frac{1}{|W|}\int_T f(t)|\Delta(t)|^2 dt$$

2. Proof of the Weyl Character Formula

2.1. Assume now that $\pi \in \widehat{G}$ is an irreducible representation of highest weight $\lambda \in \Lambda^+$. From the orthogonality of characters, we have

$$\int_G |\chi_\lambda(g)|^2 dg = 1$$

Since χ_λ is a class-function, the integration formula yields

$$\int_T |\chi_\lambda(t)\Delta(t)|^2 dt = |W|$$

2.2. We now analyze the integrand $\chi_\lambda\Delta$. First, note that since $W \simeq N_G(T)/T$, there exists $n \in N_G(T)$ such that $\pi(n) : V(\lambda) \to V(w\lambda)$. Example: if $w = s_{12}$, then $n = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ [homework]. In particular, $\dim V(\mu) = \dim V(w\mu)$. Hence, if $n_\mu = \dim V(\mu)$, then

$$\chi_\lambda|_T = \sum n_\mu e^\mu \quad \text{and} \quad n_{w\mu} = n_\mu, \quad \forall \mu \in \mathcal{E}(\pi), \forall w \in W$$

On the other hand since Δ is W-skew symmetric it then $\chi_\lambda\Delta$ is W-skew symmetric. This means that if we write

$$\chi_\lambda\Delta = (\sum_\mu n_\mu e^\mu)(\sum_{w\in W}\epsilon(w)e^{w\rho}) = \sum c(\beta)e^\beta$$

where $c(\beta) \in \mathbb{Z}$ are the coefficients of the various $\beta = w\rho + \mu$ after we open the parenthesis, then

$$c(w\beta) = \epsilon(w)c(\beta), \quad \forall w \in W$$

Since $c(\lambda + \rho) = 1$, we have $c(w(\lambda + \rho)) = \epsilon(w)$, $\forall w \in W$.

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[1] The details of this computation are presented in Fulton and Harris, *Representation theory*, p. 443.

On the other hand, from Parseval identity on the torus

$$\int_T |\chi_\lambda \Delta|^2 = \sum_\beta |c(\beta)|^2$$

implies

$$\begin{aligned}|W| = \sum_\beta |c(\beta)|^2 &= \sum_{w \in W} |c(w(\lambda+\rho))|^2 + \sum_{\beta \notin W\cdot(\lambda+\rho)} |c(\beta)|^2 \\ &= 1 + 1 + \cdots + 1 \ (|W| \text{ times}) + \sum_{\beta \notin W\cdot(\lambda+\rho)} |c(\beta)|^2\end{aligned}$$

Hence $c(\beta) = 0$ when $\beta \notin W \cdot (\lambda+\rho)$ and $\chi_\lambda \Delta = \sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)}$.

3. DIMENSION FORMULA

3.1. **Example:** $SU(2)$**.** In the case $G = SU(2)$, the representation σ_m of highest weight $m \geq 0$ has trace

$$\chi_m\Big(\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{bmatrix}\Big) = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

Hence we can obtain a formula for $\dim \sigma_m$ using L'Hopital

$$\dim \sigma_m = \chi_m(I_2) = \lim_{\theta\to 0} \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = m+1$$

We will use this idea in a more general context to prove the following

3.2. **Theorem.** Assume π_λ is the representation of G of highest weight $\lambda \in \Lambda^+$. Then

$$\dim \pi_\lambda = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)}$$

Proof. Let $H_0 \in \mathfrak{t}$ such that $\mu(H_0) = (\mu, \rho)$, $\forall \mu \in i\mathfrak{t}^*$. Such a H_0 exists and is unique (since the inner product is definite). Then

$$\begin{aligned}\dim \pi_\lambda = \chi_\lambda(e) = \lim_{t\to 0} \chi_\lambda(\exp(tH_0)) &= \lim_{t\to 0} \prod_{\alpha\in\Phi^+} \frac{\sum_{w\in W} \epsilon(w) e^{w(\rho+\lambda)(tH_0)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2(tH_0)} - e^{-\alpha/2(tH_0)})} \\ = \lim_{t\to 0} \prod_{\alpha\in\Phi^+} \frac{\sum_{w\in W}\epsilon(w) e^{t(w(\rho+\lambda),\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\frac{t}{2}(\alpha,\rho)} - e^{-\frac{t}{2}(\alpha,\rho)})}\end{aligned}$$

The numerator of this expression can be re-written (since W preserves the inner product) as

$$\sum_{w\in W} \epsilon(w) e^{t(w\rho, \rho+\lambda)} = \prod_{\alpha\in\Phi^+} (e^{\frac{t}{2}(\alpha,\rho+\lambda)} - e^{-\frac{t}{2}(\alpha,\rho+\lambda)})$$

by using the identity for Δ. Hence

$$\dim \pi_\lambda = \lim_{t\to 0} \prod_{\alpha\in\Phi^+} \frac{e^{\frac{t}{2}(\alpha,\rho+\lambda)} - e^{-\frac{t}{2}(\alpha,\rho+\lambda)}}{e^{\frac{t}{2}(\alpha,\rho)} - e^{-\frac{t}{2}(\alpha,\rho)}}$$

By L'Hopital, the term corresponding to $\alpha \in \Phi^+$ tends to $\frac{(\alpha,\lambda+\rho)}{(\alpha,\rho)}$. This finishes the proof.

3.3. **Example:** $SU(3)$**.** Here $\rho = l_1 - l_3$ and for $\lambda = ml_1 - nl_2$, we have

$$(\lambda, \alpha_{12}) = m, \quad (\rho, \alpha_{12}) = 1, \quad (\lambda, \rho_{23}) = n+1, \quad (\rho, \alpha_{23}) = 1, \quad (\lambda, \rho_{13}) = m+n, \quad (\rho, \alpha_{13}) = 2$$

hence

$$\dim \sigma_{m,n} = \prod_{i<j} \frac{(\lambda+\rho, \alpha_{ij})}{(\rho, \alpha_{ij})} = \frac{(m+1)(n+1)(m+n+2)}{2}$$