12. WEYL’S CHARACTER FORMULA

1. Weyl’s Integration formula

1.1. Set-up. $G$ is a compact, connected Lie group, $T$ a fixed maximal torus. The Haar measures $dg$ on $G$ and $dt$ on $T$ are normalized so that $\text{vol}(G) = \int_G dg = 1$ and $\text{vol}(T) = \int_T dt = 1$. Also, the coset space $G/T$ is endowed with a left-invariant (under the $G$-action) measure $d\mu_{G/T}$ and we normalized it so that $\text{vol}(G/T) = \int_{G/T} d\mu_{G/T} = 1$. Let $\Delta = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W} e^{w\rho}$ the denominator in Weyl’s character formula. With the above normalizations we have the following

1.2. Theorem. Assume $f \in C(G)$ is a continuous function on $G$. Then

$$\int_G f(g)dg = \frac{1}{|W|} \int_{G/T} \int_T f(gtg^{-1})|\Delta(t)|^2 dt d\mu_{G/T}(gT)$$

We skip over the details of this proof. The formula is obtained by applying the change of variable formula to the map

$$\alpha : G/T \times T \rightarrow G, \quad \alpha(gT, h) = ghg^{-1}$$

The idea is that this function is surjective, since every element of $G$ has a conjugate in $T$. Moreover (outside a negligible set) $\mu$ is a covering map whose fiber is $N_G(T)/T$. Part of the theory is to establish the bijection $N_G(T)/T \simeq W$. In the case of $SU(3)$ this is not hard to see since both can be identified with $S_3$. Hence the size of (almost all) fibers of $\alpha$ is $|W|$, and this explains the factor $\frac{1}{|W|}$ on the right-hand side. Then the factor $|\Delta|^2$ is simply the Jacobian of $\mu$.

1.3. Corollary. If $f \in C(G)_{\text{class}}$ is a class function, that is $f(x) = f(y), \forall x, y \in G$, then

$$\int_G f(g)dg = \frac{1}{|W|} \int_T f(t)|\Delta(t)|^2 dt$$

2. Proof of the Weyl Character Formula

2.1. Assume now that $\pi \in \hat{G}$ is an irreducible representation of highest weight $\lambda \in \Lambda^+$. From the orthogonality of characters, we have

$$\int_G |\chi_\lambda(g)|^2 dg = 1$$

Since $\chi_\lambda$ is a class-function, the integration formula yields

$$\int_T |\chi_\lambda(t)\Delta(t)|^2 dt = |W|$$

2.2. We now analyze the integrand $\chi_\lambda \Delta$. First, note that since $W \simeq N_G(T)/T$, there exists $n \in N_G(T)$ such that $\pi(n) : V(\lambda) \rightarrow V(w\lambda)$. Example: if $w = s_{12}$, then $n = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ [homework]. In particular, $\dim V(\mu) = \dim V(w\mu)$. Hence, if $n_\mu = \dim V(\mu)$, then

$$\chi_\lambda|_T = \sum n_\mu e^\mu \quad \text{and} \quad n_{w\mu} = n_\mu, \quad \forall \mu \in \mathcal{E}(\pi), \forall w \in W$$

On the other hand since $\Delta$ is $W$-skew symmetric it then $\chi_\lambda \Delta$ is $W$-skew symmetric. This means that if we write

$$\chi_\lambda \Delta = \left( \sum_{\mu} n_\mu e^\mu \right) \left( \sum_{w \in W} e(w) e^{w\rho} \right) = \sum_{\beta} c(\beta) e^\beta$$

where $c(\beta) \in \mathbb{Z}$ are the coefficients of the various $\beta = w\rho + \mu$ after we open the parenthesis, then

$$c(w\beta) = c(w) c(\beta), \quad \forall w \in W$$

Since $c(\lambda + \rho) = 1$, we have $c(w(\lambda + \rho)) = c(w), \forall w \in W$.

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1The details of this computation are presented in Fulton and Harris, Representation theory, p. 443.
On the other hand, from Parseval identity on the torus
\[ \int_T |\chi_\lambda \Delta|^2 = \sum_\beta |c(\beta)|^2 \]
implies
\[ |W| = \sum_\beta |c(\beta)|^2 = \sum_{w \in W} |c(w(\lambda + \rho))|^2 + \sum_{\beta \notin W \cdot (\lambda + \rho)} |c(\beta)|^2 \]
\[ = 1 + 1 + \cdots + 1 (|W| \text{ times}) + \sum_{\beta \notin W \cdot (\lambda + \rho)} |c(\beta)|^2 \]
Hence \( c(\beta) = 0 \) when \( \beta \notin W \cdot (\lambda + \rho) \) and \( \chi_\lambda \Delta = \sum_{w \in W} e(w)e^{w(\lambda + \rho)}. \)

3. Dimension formula

3.1. Example: SU(2). In the case \( G = SU(2) \), the representation \( \sigma_m \) of highest weight \( m \geq 0 \) has trace
\[ \chi_m \left( \begin{bmatrix} e^{i\theta} & e^{-i\theta} \end{bmatrix} \right) = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} \]
Hence we can obtain a formula for \( \dim \sigma_m \) using L'Hôpital
\[ \dim \sigma_m = \chi_m(I_2) = \lim_{\theta \to 0} \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = m + 1 \]
We will use this idea in a more general context to prove the following

3.2. Theorem. Assume \( \pi_\lambda \) is the representation of \( G \) of highest weight \( \lambda \in \Lambda^+ \). Then
\[ \dim \pi_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{\rho, \alpha} \]

Proof. Let \( H_0 \in \mathfrak{t} \) such that \( \mu(H_0) = (\mu, \rho), \forall \mu \in \mathfrak{t}^* \). Such a \( H_0 \) exists and is unique (since the inner product is definite). Then
\[ \dim \pi_\lambda = \chi_\lambda(e) = \lim_{t \to 0} \chi_\lambda(\exp(tH_0)) = \lim_{t \to 0} \prod_{\alpha \in \Phi^+} \frac{\sum_{w \in W} e(w)e^{w(\rho + \lambda)(tH_0)}}{\prod_{\alpha \in \Phi^+} (e^{\rho, \alpha}/2(tH_0) - e^{-\rho, \alpha}/2(tH_0))} \]
\[ = \lim_{t \to 0} \prod_{\alpha \in \Phi^+} \frac{\sum_{w \in W} e(w)e^{w(\rho + \lambda)}}{\prod_{\alpha \in \Phi^+} (e^{\rho, \alpha}/2 + e^{-\rho, \alpha})} \]
The numerator of this expression can be re-written (since \( W \) preserves the inner product) as
\[ \sum_{w \in W} e(w)e^{w(\rho + \lambda)} = \prod_{\alpha \in \Phi^+} (e^{\rho, \alpha}/2 + e^{-\rho, \alpha}) \]
by using the identity for \( \Delta \). Hence
\[ \dim \pi_\lambda = \lim_{t \to 0} \prod_{\alpha \in \Phi^+} \frac{e^{\rho, \alpha}/2 + e^{-\rho, \alpha}}{e^{\rho, \alpha}/2 - e^{-\rho, \alpha}} \]
By L'Hôpital, the term corresponding to \( \alpha \in \Phi^+ \) tends to \( \frac{(\alpha, \lambda + \rho)}{(\alpha, \rho)} \). This finishes the proof.

3.3. Example: SU(3). Here \( \rho = l_1 - l_3 \) and for \( \lambda = ml_1 - nl_2, \) we have
\[ (\lambda, \alpha_{12}) = m, \quad (\rho, \alpha_{12}) = 1, \quad (\lambda, \rho_{23}) = n + 1, \quad (\rho, \rho_{23}) = 1, \quad (\lambda, \rho_{13}) = m + n, \quad (\rho, \alpha_{13}) = 2 \]
hence
\[ \dim \sigma_{m,n} = \prod_{i<j} \frac{(\lambda + \rho, \alpha_{ij})}{\rho, \alpha_{ij}} = \frac{(m+1)(n+1)(m+n+2)}{2} \]