2. GROUP REPRESENTATIONS

1. Vector spaces

1.1. **Definitions.**

1.1.1. *Notation.* $F =$ field $(= \mathbb{R}, \mathbb{C})$, $V =$ n-dimensional vector space over F.

1.1.2. *Examples.* $V = \mathbb{C}^3, F = \mathbb{C}$, $V = \mathbb{C}^3, F = \mathbb{R}$.

1.1.3. *Dimension.* In general, there is an isomorphism: $V \simeq F^n$, where $n = \dim_F V$.

1.1.4. *Note.* For a given vector space V as above, the isomorphism onto F^n depends on the choice of basis. For example, the 2-dimensional complex space

$$U = \{(x_1, \ldots, x_3) : x_1 + x_2 + x_3 = 0\}$$

doesn't have a "standard" basis, so we don't have a natural identification with $\mathbb{C}^2$. We can turn this to our advantage by thinking of U as yet another realization of $\mathbb{C}^2$.

1.1.5. *Subspace.* Notation: $U \leq_F \mathbb{C}^3$.

1.2. **Linear maps.**

1.2.1. $\mathcal{L}(V, W) := \{F-\text{linear maps } \ T : V \to W\}$. Then $\mathcal{L}(V, W)$ is a vector space over F on its own, of dimension $\dim_F V \times \dim_F W$. Also, $\mathcal{L}(V) := \mathcal{L}(V, V)$. If the underlying field needs to be specified one uses the notation $\mathcal{L}_F(V, W)$.

1.2.2. Once we choose a basis $\mathcal{B}$ on V we have an isomorphism $\Phi_{\mathcal{B}} : \mathcal{L}(V) \simeq \mathcal{M}_{n\times n}(n, F)$. For a different choice of basis $\mathcal{B}'$, we have $\Phi_{\mathcal{B}'} = C\Phi_{\mathcal{B}}C^{-1}$, where $C = C_{\mathcal{B}\to\mathcal{B}'}$ is the transition matrix.

1.3. **Invariants.** $\mathrm{tr}, \det : \mathcal{L}(V) \to F$. det is multiplicative, and $GL(V) = \{T \in \mathcal{L}(V) : \det T \neq 0\}$.

1.4. $GL(V) = \{\text{invertible linear maps } T : V \to V\} \subset \mathcal{L}(V)$. Note that $GL(V)$ is a group, not a vector space. Once we choose a basis $\mathcal{B}$, the restriction of $\Phi_{\mathcal{B}}$ to $GL(V)$ determines a group isomorphism $GL(V) \simeq GL(n, F)$.

2. Representations

2.1. **Definition.** $G =$ group, $V =$ vector space over F. A representation of G on V is an action of G on V by F-linear maps. In other words, a group homomorphism

$$\pi : G \to GL(V)$$

Each $g \in G$ acts on V via the linear operator $\pi(g) : V \to V$. From now on, we will only talk about $F = \mathbb{C}$ and hence look at complex representations.

2.1.1. *Examples.* Trivial representation. $\pi : G \to GL(V), \pi(g) = Id_V$.

Standard representation. $id : GL(V) \to GL(V)$.

2.2. **Group characters.**

2.2.1. *Definition.* One-dimensional representation correspond to group characters, that is group homomorphisms

$$\chi : G \to \mathbb{C}^\times$$

The set of group characters over G form an abelian group. We denote it henceforth by G^*.

2.2.2. *Example:* $\mathbb{Z}/N\mathbb{Z}$. Since this is a cyclic group, a homomorphism $\chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$ is uniquely determined by $\chi(1)$, its value on the generator. Since $1 = \chi(0) = \chi(1 + \cdots + 1) = \chi(1)^N$ it means that $\chi(1)$ is an N^{th} root of unity. Since these are $1, \zeta_N, \ldots, \zeta_N^{N-1}$ (with ζ_N a primitive root), we can label the group characters of $\mathbb{Z}/N\mathbb{Z}$ by $\chi_j(a) = \zeta_N^j$, $0 \leq j \leq N - 1$. In this case we notice that $G^* \simeq G$, and this is a general feature of finite abelian groups [homework].

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2.3. **Intertwining operator.** Given two representations (π, V) and (σ, W) of the same group G, an *intertwining operator* is a linear map $T : V \to W$ with the following property

$$T(\pi(g)v) = \sigma(g)(T(v)), \quad \forall\, g \in G, v \in V$$

The set of all intertwining operators is clearly a (complex) vector space (possibly zero) and we call it $\mathrm{Hom}_G(V, W)$.

2.4. **Equivalence of representations.** Two representations (π, V) and (σ, W) of the same group G are said equivalent if there exists an intertwining operator $T \in \mathrm{Hom}_G(V, W)$ which is a linear isomorphism $T : V \simeq W$. This is an equivalence relation. We use the notation $\pi \simeq_G \sigma$ or simply $\pi \simeq \sigma$ if the group is understood.

2.4.1. *Simple case.* In the case of one-dimensional representations (group characters), two such representations are equivalent if and only if they are equal.

2.5. **Irreducible Representations.**

2.5.1. *Definition.* Assume (π, V) is a representation of G. A linear subspace $W \leq V$ is G-invariant (or G-stable) if $\pi(g)w \in W$ whenever $w \in W$ and $g \in G$. Trivially, $\{0\}$ and V itself are G-invariant.

We say that V is *irreducible* if it has no proper G-invariant subspaces.

2.5.2. *Notation.* We will use the notation $\widehat{G}$ for the set of equivalence classes of irreducible representations (of arbitrary dimension) of G.

2.6. **Cartesian Product.**

2.6.1. Assume (π, V) and (σ, W) are two representations of the same group G. Then G acts naturally on the Cartesian product $V \times W$ by $g \cdot (v, w) = (\pi(g)v, \sigma(g)w)$. This determines a representation of G and $V \times W$ and we refer to it as $\pi \times \sigma$.

2.6.2. Assume (ρ, U) is a representation of G, and assume that V, W are G-invariant, complementary subspaces of U, that is

$$U = V \oplus W, \quad V, W : \text{ both } G\text{-invariant}$$

Let π, σ denote the representations of G on V and W respectively. Then we write $\pi \oplus \sigma$ for $\pi \times \sigma$.

3. The standard representation of S_3

3.1. **Definition.** Consider the following action of S_3 on $V = \mathbb{C}^3$:

$$\pi : S_3 \to GL(\mathbb{C}^3), \quad \pi(\sigma)(x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)})$$

It is straightforward to check that this is indeed a representation. That is, $\pi(\sigma\tau) = \pi(\sigma)\pi(\tau)$, $\forall \sigma, \tau \in S_3$. We will call π the standard representation of S_3 and denote it by π_{st}.

With respect to standard basis we have the matrix representation

$$\pi_{st}((132)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{etc}$$

3.2. **Subspaces.** S_3-invariant subspaces are $L_0 = \mathbb{C}w_0$, with $w_0 = (1, 1, 1)$, and U as above. These subspaces are complementary, so $\mathbb{C}^3 =_{S_3} L_0 \oplus U$.

3.3. Check that U is irreducible: there are no invariant vectors (up to scalar). If we choose any basis e, f for U, then with respect to the basis $\mathcal{B} = \{w_0, e, f\}$, each $\pi(g)$ is block diagonal of the type $\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$.

3.4. **Restriction to A_3.** As a subgroup of S_3, the group $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$ also acts on U. As a representation of A_3, U is reducible and can be written as $U = L_1 \oplus L_2$, where $L_1 = \mathbb{C}e$ and $L_2 = \mathbb{C}f$ are orthogonal one-dimensional A_3-invariant subspaces (check). Then with respect to $\{w_0, e, f\}$, all $\pi_{st}(g)$, for $g \in A_3$ is given by a diagonal matrix

$$\pi(g)_{\mathcal{B}} = \begin{bmatrix} \chi_0(g) & 0 & 0 \\ 0 & \chi_1(g) & 0 \\ 0 & 0 & \chi_2(g) \end{bmatrix}$$

where $\chi_0, \chi_1, \chi_2 : A_3 \to C^\times$ are group characters. That means that we have an equivalence of A_3-representations: $\mathbb{C}^3 \simeq_{A_3} \chi_0 \oplus \chi_1 \oplus \chi_2$.

4. Abelian groups

4.1. **Theorem.** For finite abelian groups, irreducible representations occur only in dimension one. In other words, $\widehat{G} = G^*$.

4.1.1. *Proof.* Assume (π, V) is an irreducible representation of G (abelian). If $g \in G$ and $\lambda \in \mathbb{C}$ is an eigenvalue of $\pi(g)$, then $V_\lambda = \{v \in V : \pi(g_1)v = \lambda v\}$ is G-invariant and non-zero, hence $V_\lambda = V$. Consequently, all operators $\pi(g)$, $g \in G$ are scalar multiples of the identity. This implies that for $v \neq 0$, $\mathbb{C} \cdot v$ is G-invariant. Hence $V = \mathbb{C} \cdot v$, that is $\dim_{\mathbb{C}} V = 1$.

5. Character of a representation

5.1. **Definition.** Assume (π, V) is a representation (not necessarily irreducible) of the finite group G. We define the *character of the representation* π, denoted χ_π, by

$$\chi_\pi(g) = \mathrm{trace}(\pi(g))$$

Notice that χ_π is conjugation invariant, that is

$$\chi_\pi(xyx^{-1}) = \chi_\pi(x), \quad \forall x, y \in G$$

5.1.1. *Note.* χ_π is not a *group character* (that is $\chi_\pi(xy) \neq \chi_\pi(x)\chi_\pi(y)$) but simply a function $\chi_\pi : G \to \mathbb{C}$.

6. Regular Representation

6.1. **L^2 inner product.** We denote by $L^2(G)$ the set of complex-valued functions $\phi : G \to \mathbb{C}$. This linear vector space is endowed with the standard L^2-inner product

$$< \phi, \psi >_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g}, \quad \|\phi\|^2 = \frac{1}{|G|} \sum_{g \in G} |\phi(g)|^2$$

6.2. **Left and right actions.** For $g \in G$ and $\phi \in L^2(G)$, we define

$$L(g)\phi : G \to \mathbb{C}, \quad L(g)\phi(x) = \phi(g^{-1}x), \ \forall x \in G$$

$$R(g)\phi : G \to \mathbb{C}, \quad R(g)\phi(x) = \phi(xg), \ \forall x \in G$$

It is immediate to check that $(L, L^2(G))$ and $(R, L^2(G))$ are *unitary* representations of G, in that

$$L(gh) = L(g)L(h), \quad \|L(g)\phi\| = \|\phi\|, \ \forall g, h \in G, \ \forall \phi \in L^2(G)$$

$$R(gh) = R(g)R(h), \quad \|R(g)\phi\| = \|\phi\|, \ \forall g, h \in G, \ \forall \phi \in L^2(G)$$

These two are called the left-regular and resp. right-regular representation.

6.2.1. It is easy to check that the two actions commute

$$L(g)R(h) = R(h)L(g), \quad \forall g, h \in G$$

In particular, this allows us to define an action of $G \times G$ on $L^2(G)$ given by

$$L \times R(g, h)\phi(x) = L(g)R(h)\phi(x) = \phi(g^{-1}xh)$$

7. Contragredient Representation

7.1. **Definition.** Assume (π, V) is a representation of the group G. Consider the dual space:

$$V^* = \mathcal{L}(V) = \{l : V \to \mathbb{C} \text{ linear map}\}$$

Then π induces an action of G on V^*, call it π^*, given by

$$\pi^*(g)(l)(v) = l\big(\pi(g^{-1})v\big), \quad g \in G, l \in V^*, v \in V$$

Then π^* is called the *contragredient* representation (of π). It is straightforward to check [homework] that

$$\chi_{\pi^*}(g) = \overline{\chi_\pi(g)}, \quad \forall g \in G$$

In general, π and π^* are not equivalent (they have different character) unless the character of π is real-valued [we will prove this later].