2. GROUP REPRESENTATIONS

1. Vector spaces

1.1. Definitions.

1.1.1. Notation. \( F = \text{field } (= \mathbb{R}, \mathbb{C}), \) \( V = \text{n-dimensional vector space over } F. \)

1.1.2. Examples. \( V = \mathbb{C}^3, F = \mathbb{C}, V = \mathbb{C}^3, F = \mathbb{R}. \)

1.1.3. Dimension. In general, there is an isomorphism: \( V \cong F^n, \) where \( n = \dim_F V. \)

1.1.4. Note. For a given vector space \( V \) as above, the isomorphism onto \( F^n \) depends on the choice of basis. For example, the 2-dimensional complex space
\[
U = \{(x_1, \ldots, x_3) : x_1 + x_2 + x_3 = 0\}
\]
doesn’t have a "standard" basis, so we don’t have a natural identification with \( \mathbb{C}^2. \) We can turn this to our advantage by thinking of \( U \) as yet another realization of \( \mathbb{C}^2. \)

1.1.5. Subspace. Notation: \( U \leq_F \mathbb{C}^3. \)

1.2. Linear maps.

1.2.1. \( \mathcal{L}(V, W) := \{ F - \text{linear maps } T : V \to W \}. \) Then \( \mathcal{L}(V, W) \) is a vector space over \( F \) on its own, of dimension \( \dim_F V \times \dim_F W. \) Also, \( \mathcal{L}(V) := \mathcal{L}(V, V). \) If the underlying field needs to be specified one uses the notation \( \mathcal{L}_F(V, W). \)

1.2.2. Once we choose a basis \( \mathcal{B} \) on \( V \) we have an isomorphism \( \Phi_\mathcal{B} : \mathcal{L}(V) \cong M_{n \times n}(n, F). \) For a different choice of basis \( \mathcal{B}', \) we have \( \Phi_\mathcal{B}' = C\Phi_\mathcal{B}C^{-1}, \) where \( C = C_{\mathcal{B} \to \mathcal{B}'} \) is the transition matrix.

1.3. Invariants. \( \text{tr}, \det : \mathcal{L}(V) \to F. \) \( \text{tr} \) is multiplicative, and \( GL(V) = \{ T \in \mathcal{L}(V) : \det T \neq 0 \}. \)

1.4. \( GL(V) = \{ \text{invertible linear maps } T : V \to V \} \subset \mathcal{L}(V). \) Note that \( GL(V) \) is a group, not a vector space. Once we choose a basis \( \mathcal{B}, \) the restriction of \( \Phi_\mathcal{B} \) to \( GL(V) \) determines a group isomorphism \( GL(V) \cong GL(n, F). \)

2. Representations

2.1. Definition. \( G = \text{group}, V = \text{vector space over } F. \) A representation of \( G \) on \( V \) is an action of \( G \) on \( V \) by \( F \)-linear maps. In other words, a group homomorphism
\[
\pi : G \to GL(V)
\]
Each \( g \in G \) acts on \( V \) via the linear operator \( \pi(g) : V \to V. \) From now on, we will only talk about \( F = \mathbb{C} \) and hence look at complex representations.

2.1.1. Examples. Trivial representation. \( \pi : G \to GL(V), \pi(g) = Id_V. \)
Standard representation. \( id : GL(V) \to GL(V). \)

2.2. Group characters.

2.2.1. Definition. One-dimensional representation correspond to group characters, that is group homomorphisms
\[
\chi : G \to \mathbb{C}^x
\]
The set of group characters over \( G \) form an abelian group. We denote it henceforth by \( G^*. \)

2.2.2. Example: \( \mathbb{Z}/NZ. \) Since this is a cyclic group, a homomorphism \( \chi : \mathbb{Z}/NZ \to \mathbb{C}^x \) is uniquely determined by \( \chi(1), \) its value on the generator. Since \( 1 = \chi(0) = \chi(1 + \cdots + 1) = \chi(1)^N \) it means that \( \chi(1) \) is an \( N^{th} \) root of unity. Since these are \( 1, \zeta_N, \ldots, \zeta_N^{N-1} \) (with \( \zeta_N \) a primitive root), we can label the group characters of \( \mathbb{Z}/NZ \) by \( \chi_j(a) = \zeta_N^j, \) \( 0 \leq j \leq N - 1. \) In this case we notice that \( G^* \cong G, \) and this is a general feature of finite abelian groups [homework].

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2.3. Intertwining operator. Given two representations \((\pi, V)\) and \((\sigma, W)\) of the same group \(G\), an intertwining operator is a linear map \(T: V \to W\) with the following property
\[ T(\pi(g)v) = \sigma(g)(T(v)), \quad \forall g \in G, v \in V \]
The set of all intertwining operators is clearly a (complex) vector space (possibly zero) and we call it \(\text{Hom}_G(V, W)\).

2.4. Equivalence of representations. Two representations \((\pi, V)\) and \((\sigma, W)\) of the same group \(G\) are said equivalent if there exists an intertwining operator \(T \in \text{Hom}_G(V, W)\) which is a linear isomorphism \(T: V \cong W\). This is an equivalence relation. We use the notation \(\pi \cong_G \sigma\) or simply \(\pi \cong \sigma\) if the group is understood.

2.4.1. Simple case. In the case of one-dimensional representations (group characters), two such representations are equivalent if and only if they are equal.

2.5. Irreducible Representations.

2.5.1. Definition. Assume \((\pi, V)\) is a representation of \(G\). A linear subspace \(W \leq V\) is \(G\)-invariant (or \(G\)-stable) if \(\pi(g)w \in W\) whenever \(w \in W\) and \(g \in G\). Trivially, \(\{0\}\) and \(V\) itself are \(G\)-invariant.

We say that \(V\) is irreducible if it has no proper \(G\)-invariant subspaces.

2.5.2. Notation. We will use the notation \(\hat{G}\) for the set of equivalence classes of irreducible representations (of arbitrary dimension) of \(G\).

2.6. Cartesian Product.

2.6.1. Assume \((\pi, V)\) and \((\sigma, W)\) are two representations of the same group \(G\). Then \(G\) acts naturally on the Cartesian product \(V \times W\) by \(g \cdot (v, w) = (\pi(g)v, \sigma(g)w)\). This determines a representation of \(G\) and \(V \times W\) and we refer to it as \(\pi \times \sigma\).

2.6.2. Assume \((\rho, U)\) is a representation of \(G\), and assume that \(V, W\) are \(G\)-invariant, complementary subspaces of \(U\), that is
\[ U = V \oplus W; \quad V, W: \text{both } G\text{-invariant} \]
Let \(\pi, \sigma\) denote the representations of \(G\) on \(V\) and \(W\) respectively. Then we write \(\pi \oplus \sigma\) for \(\pi \times \sigma\).

3. The standard representation of \(S_3\)

3.1. Definition. Consider the following action of \(S_3\) on \(V = \mathbb{C}^3\):
\[ \pi: S_3 \to GL(\mathbb{C}^3), \quad \pi(\sigma)(x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}) \]
It is straightforward to check that this is indeed a representation. That is, \(\pi(\sigma \tau) = \pi(\sigma)\pi(\tau), \forall \sigma, \tau \in S_3\).
We will call \(\pi\) the standard representation of \(S_3\) and denote it by \(\pi_{st}\).

With respect to standard basis we have the matrix representation
\[ \pi_{st}((132)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]
etc

3.2. Subspaces. \(S_3\)-invariant subspaces are \(L_0 = \mathbb{C}w_0\), with \(w_0 = (1, 1, 1)\), and \(U\) as above. These subspaces are complementary, so \(\mathbb{C}^3 = S_3 L_0 \oplus U\).

3.3. Check that \(U\) is irreducible: there are no invariant vectors (up to scalar). If we choose any basis \(e, f\) for \(U\), then with respect to the basis \(\mathcal{B} = \{w_0, e, f\}\), each \(\pi(g)\) is block diagonal of the type
\[ \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \end{bmatrix} \text{ etc} \]
3.4. **Restriction to** $A_3$. As a subgroup of $S_3$, the group $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$ also acts on $U$. As a representation of $A_3$, $U$ is reducible and can be written as $U = L_1 \oplus L_2$, where $L_1 = \mathbb{C}e$ and $L_2 = \mathbb{C}f$ are orthogonal one-dimensional $A_3$-invariant subspaces (check). Then with respect to $\{w_0,e,f\}$, all $\pi_{\alpha}(g)$, for $g \in A_3$ is given by a diagonal matrix

$$\pi(g)_B = \begin{bmatrix} \chi_0(g) & 0 & 0 \\ 0 & \chi_1(g) & 0 \\ 0 & 0 & \chi_2(g) \end{bmatrix}$$

where $\chi_0, \chi_1, \chi_2 : A_3 \to \mathbb{C}^\times$ are group characters. That means that we have an equivalence of $A_3$-representations: $\mathbb{C}^3 \simeq_{A_3} \chi_0 \oplus \chi_1 \oplus \chi_2$.

4. **Abelian groups**

4.1. **Theorem.** For finite abelian groups, irreducible representations occur only in dimension one. In other words, $\hat{G} = G^*$.

4.1.1. **Proof.** Assume $(\pi, V)$ is an irreducible representation of $G$ (abelian). If $g, h \in G$ and $\lambda \in \mathbb{C}$ is an eigenvalue of $\pi(g)$, then $V_\lambda = \{v \in V : \pi(g)v = \lambda v\}$ is $G$-invariant and non-zero, hence $V_\lambda = V$. Consequently, all operators $\pi(g)$, $g \in G$ are scalar multiples of the identity. This implies that for $v \neq 0$, $\mathbb{C} \cdot v$ is $G$-invariant. Hence $V = \mathbb{C} \cdot v$, that is $\dim_{\mathbb{C}} V = 1$.

5. **Character of a representation**

5.1. **Definition.** Assume $(\pi, V)$ is a representation (not necessarily irreducible) of the finite group $G$. We define the character of the representation $\pi$, denoted $\chi_\pi$, by

$$\chi_\pi(g) = \text{trace}(\pi(g))$$

Notice that $\chi_\pi$ is conjugation invariant, that is

$$\chi_\pi(xy^{-1}) = \chi_\pi(x), \quad \forall x, y \in G$$

5.1.1. **Note.** $\chi_\pi$ is not a group character (that is $\chi_\pi(xy) \neq \chi_\pi(x)\chi_\pi(y)$) but simply a function $\chi_\pi : G \to \mathbb{C}$.

6. **Regular Representation**

6.1. **$L^2$ inner product.** We denote by $L^2(G)$ the set of complex-valued functions $\phi : G \to \mathbb{C}$. This linear vector space is endowed with the standard $L^2$-inner product

$$\langle \phi, \psi \rangle_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}, \quad \|\phi\|^2 = \frac{1}{|G|} \sum_{g \in G} |\phi(g)|^2$$

6.2. **Left and right actions.** For $g \in G$ and $\phi \in L^2(G)$, we define

$$L(g)\phi : G \to \mathbb{C}, \quad L(g)\phi(x) = \phi(g^{-1}x), \quad \forall x \in G$$

$$R(g)\phi : G \to \mathbb{C}, \quad R(g)\phi(x) = \phi(xg), \quad \forall x \in G$$

It is immediate to check that $(L, L^2(G))$ and $(R, L^2(G))$ are unitary representations of $G$, in that

$$L(gh) = L(g)L(h), \quad \|L(g)\phi\| = \|\phi\|, \quad \forall g, h \in G, \forall \phi \in L^2(G)$$

$$R(gh) = R(g)R(h), \quad \|R(g)\phi\| = \|\phi\|, \quad \forall g, h \in G, \forall \phi \in L^2(G)$$

These two are called the left-regular and resp. right-regular representation.

6.2.1. It is easy to check that the two actions commute

$$L(g)R(h) = R(h)L(g), \quad \forall g, h \in G$$

In particular, this allows us to define an action of $G \times G$ on $L^2(G)$ given by

$$L \times R(g, h)\phi(x) = L(g)R(h)\phi(x) = \phi(g^{-1}xh)$$
7. CONTRAGREDENT REPRESENTATION

7.1. **Definition.** Assume \((\pi, V)\) is a representation of the group \(G\). Consider the dual space:

\[ V^* = \mathcal{L}(V) = \{ l : V \to \mathbb{C} \text{ linear map} \} \]

Then \(\pi\) induces an action of \(G\) on \(V^*\), call it \(\pi^*\), given by

\[ \pi^*(g)(l)(v) = l(\pi(g^{-1})v), \quad g \in G, l \in V^*, v \in V \]

Then \(\pi^*\) is called the **contragredient** representation (of \(\pi\)). It is straightforward to check [homework] that

\[ \chi_{\pi^*}(g) = \overline{\chi_{\pi}(g)}, \quad \forall g \in G \]

In general, \(\pi\) and \(\pi^*\) are not equivalent (they have different character) unless the character of \(\pi\) is real-valued [we will prove this later].