

4. ORTHOGONALITY RELATIONS

1. ORTHOGONALITY RELATIONS

1.1. **Matrix coefficients.** Assume (π, V) is a unitary representation of G . Then, for $v, w \in V$, the following function

$$c_{vw} : G \rightarrow \mathbb{C}, \quad c_{vw}(g) = \langle \pi(g)v, w \rangle$$

is called a *matrix coefficient function* associated to (vectors in) V . The following theorem (due to Schur) computes the inner product of matrix coefficients coming from different (irreducible) representations.

1.2. **Theorem.** Let (π_1, V_1) and (π_2, V_2) two irreducible representations of G , and let $v_1, w_1 \in V_1$ and $v_2, w_2 \in V_2$ arbitrary vectors. Then:

$$\langle c_{v_1 w_1}, c_{v_2 w_2} \rangle_{L^2(G)} = \begin{cases} 0, & V_1 \not\cong V_2 \\ \frac{1}{\dim V} \langle v_1, v_2 \rangle \overline{\langle w_1, w_2 \rangle}, & V_1 = V_2 \end{cases}$$

1.2.1. *Proof.* The LHS of the identity is

$$\langle c_{v_1 w_1}, c_{v_2 w_2} \rangle = \frac{1}{|G|} \sum_{g \in G} c_{v_1 w_1}(g) \overline{c_{v_2 w_2}(g)} = \frac{1}{|G|} \sum_{g \in G} \langle \pi_1(g)v_1, w_1 \rangle \overline{\langle \pi_2(g)v_2, w_2 \rangle}$$

This is clearly skew-bilinear in (v_1, v_2) , and G -invariant, hence there exists an intertwining linear map (depending on w_1, w_2)

$$A_{w_1 w_2} \in \text{Hom}_G(V_1, V_2), \quad LHS = \langle A v_1, v_2 \rangle$$

By Schur's lemma, $A_{w_1 w_2} = 0$ and hence $LHS = 0$, unless V_1 and V_2 are equivalent representations.

Assume now that $V_1 = V_2 = V$. Again by Schur's lemma, there exists $\lambda_{w_1 w_2} \in \mathbb{C}$ such that $A_{w_1 w_2} = \lambda_{w_1 w_2} Id_V$. But then $\lambda_{w_1 w_2}$ is skew-bilinear in w_1, w_2 , hence there exists $B \in \text{End}_G(V)$ such that $\lambda_{w_1 w_2} = B(w_1, w_2)$. By Schur's lemma one more time, B is a scalar multiple of the identity. So far we have:

$$\langle c_{v_1 w_1}, c_{v_2 w_2} \rangle_{L^2(G)} = \lambda(V) \langle v_1, v_2 \rangle \overline{\langle w_1, w_2 \rangle}$$

To determine the constant λ , we sum over j (take the trace)

$$\frac{1}{|G|} \sum_{g \in G} \langle \pi(g)v_1, e_j \rangle \overline{\langle \pi(g)v_2, e_j \rangle} = \lambda \langle v_1, v_2 \rangle$$

to obtain $\frac{1}{|G|} \sum_{g \in G} \langle \pi(g)v_1, \pi(g)v_2 \rangle = \lambda (\dim V) \langle v_1, v_2 \rangle$. Since π is unitary, this is $\lambda = \frac{1}{\dim V}$.

2. ORTHOGONALITY OF CHARACTERS

2.1. For a representation (π, V) of G , we will denote by $\pi = \oplus m_a(\pi)a$ its decomposition into irreducible components. Then $\chi_\pi = \sum_{a \in \widehat{G}} m_a(\pi)\chi_a$.

2.1.1. *Integrated action.* For $f \in L^2(G)$, we define the operator:

$$\pi(f) : V \rightarrow V, \quad \pi(f) = \frac{1}{|G|} \sum_{g \in G} f(g)\pi(g)$$

2.2. **Theorem.** For π as above and $a, b \in \widehat{G}$ we have

- 1) $\langle \chi_a, \chi_b \rangle = \delta_{ab}$.
- 2) $\langle \chi_\pi, \chi_a \rangle = m_a(\pi)$.
- 3) $\|\chi_\pi\|_2^2 = \sum_a m_a(\pi)^2$. In particular π is irreducible if and only if $\|\chi_\pi\| = 1$.
- 4) Two representations π, π' are equivalent if and only if $\chi_\pi = \chi_{\pi'}$.
- 5) Let $\varphi_a = d_a \cdot \bar{\chi}_a$. Then $\pi(\varphi_a) : V \rightarrow V$ is the orthogonal projection onto the isotypic subspace $V(a)$.

2.2.1. *Proof.* 1) If $a \neq b$ it is clear. If $a = b$, then

$$\begin{aligned} \langle \chi_a, \chi_a \rangle &= \frac{1}{|G|} \sum_{g \in G} \left(\sum_i \langle \pi_a(g)e_i, e_i \rangle \overline{\left(\sum_j \langle \pi_a(g)e_j, e_j \rangle \right)} \right) = \sum_{i,j} \langle c_{e_i, e_i}, c_{e_j, e_j} \rangle \\ &= \frac{1}{d_a} \sum_{ij} \delta_{ij} = 1 \end{aligned}$$

2) $\langle \chi_\pi, \chi_a \rangle = \sum_b m_b(\pi) \langle \chi_b, \chi_a \rangle = \sum_b m_b(\pi) \delta_{ab} = m_a(\pi)$.

4) If $\chi_\pi = \chi_{\pi'}$ then $m_a(\pi) = \langle \chi_\pi, \chi_a \rangle = m_a(\pi')$. Then $\pi \simeq \oplus m_a(\pi)a = \oplus m_a(\pi')a \simeq \pi'$.

5) Homework exercise.

2.3. **Application.** Multiplicities in the left-regular representation: for $a \in \widehat{G}$

$$m_a(L_{reg}) = \langle \chi_{reg}, \chi_a \rangle = \frac{1}{|G|} \sum_g \chi_{reg}(g) \chi_a(g) = \frac{|G|}{|G|} \chi_a(e) = d_a$$

Since $\dim L^2(G) = |G|$ and $m_a > 0(L_{reg}) \forall a \in \widehat{G}$ it means that \widehat{G} is in fact finite, and counting the dimensions we obtain

2.4. **Theorem.** \widehat{G} is finite and $\sum_{a \in \widehat{G}} d_a^2 = |G|$.

2.5. **Examples.**

2.5.1. For the usual action of S_3 on \mathbb{C}^3 , $\chi_\pi(g) = |\text{Fix}(g)|$. Hence $\|\chi_\pi\|_2^2 = \frac{1}{|S_3|} (3^2 + \sum_{i < j} 1^2) = 2$, therefore it is the sum of two irreducible representations.

2.5.2. For the general action of S_n on \mathbb{C}^n given by $\pi(\sigma)(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$, the character is similarly given by

$$\begin{aligned} \|\chi_\pi\|^2 &= \frac{1}{|S_n|} \left\{ n^2 + \sum_{k=2}^{n-1} (n-k)^2 \cdot \#\{\text{permutations in } S_k \text{ without fixed points}\} \right\} \\ &= \sum_{k=0}^n \sum_{j=0}^k \frac{(-1)^j}{j!} \cdot \frac{(n-k)^2}{(n-k)!} = \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \cdot \frac{k^2}{k!} \end{aligned}$$

Let $w_0 = (1, 1, \dots, 1)$ and $L_0 = \mathbb{C} \cdot w_0$ the one-dimensional subspace generated by it. Since w_0 is fixed by all $g \in S_n$, L_0 is clearly S_n -invariant. Then $U = L_0^\perp = \{x_1 + \dots + x_n = 0\}$ is also invariant, and $\mathbb{C}^n = L_0 \oplus U$ is a decomposition into invariant subspaces. One can argue (just as in the case of S_3) that U is irreducible. Since π has two irreducible components the L^2 norm of its character is given by

$$\|\chi_\pi\|^2 = \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \cdot \frac{k^2}{k!} = 2$$

This identity can also be checked directly, thus giving a roundabout proof of the fact that U is irreducible.

3. PETER-WEYL THEOREM

3.1. **Notations.** Consider the following normalized matrix coefficients functions

$$\phi_{aij} : G \rightarrow \mathbb{C}, \quad \phi_{aij}(g) := d_a^{1/2} \langle e_{ai}, \pi_a(g)e_{aj} \rangle$$

where $a \in \widehat{G}$, and $\{e_{ai}\}$, $1 \leq i \leq d_a$ is an orthonormal basis for V_a .

3.2. **Theorem.** The family $\{\phi_{aij} : a \in \widehat{G}, 1 \leq i, j \leq d_a\}$ is an orthonormal basis for $L^2(G)$.

3.2.1. *Proof.* The fact that it is an orthonormal basis for ON family is a direct consequence of the orthogonality relations. The only thing we need to argue is completeness, i.e. that the set $\{\phi_{aij}\}$ spans $L^2(G)$.

Assume not. Then there exists $0 \neq f \in L^2(G)$ is such that $\langle f, \phi_{aij} \rangle = 0$, for all a, i, j . In particular this implies

$$\langle L(g)f, \phi_{aij} \rangle = 0, \quad \langle R(g)f, \phi_{aij} \rangle = 0, \quad \forall g \in G$$

To see that, note that $\langle L(g)f, \phi_{aij} \rangle = \langle f, L(g^{-1})\phi_{aij} \rangle$, and $L(g^{-1})\phi_{aij}$ is a matrix coefficient function itself [check].

Let $S \leq L^2(G)$ the linear subspace generated by $L(g)f, R(g)f$, with $g \in G$. Then S is left and right-invariant under the action of G . Let χ_S the trace of the left-action. We will see that $\langle \chi_S, \chi_a \rangle = 0$, $\forall a \in \widehat{G}$, thus arriving at a contradiction.

First note that every $F \in S$ is orthogonal on (linear combinations of) matrix coefficient functions ϕ_{aij} , in particular

$$\langle F, \chi_a \rangle = 0, \quad \forall a \in \widehat{G}$$

Consider now an orthonormal basis $\{f_\alpha\}$ of S . Then

$$\begin{aligned} \langle \chi_S, \chi_a \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_S(g) \overline{\chi_a(g)} = \frac{1}{|G|} \sum_{g \in G} \left\{ \sum_{\alpha} \langle L(g)f_\alpha, f_\alpha \rangle \right\} \overline{\chi_a(g)} \\ &= \frac{1}{|G|^2} \sum_{g \in G} \sum_{\alpha} \sum_{h \in G} f_\alpha(g^{-1}h) \overline{f_\alpha(h)} \overline{\chi_a(g)} = \frac{1}{|G|^2} \sum_{g \in G} \sum_{\alpha} \sum_{h \in G} f_\alpha(h) \overline{f_\alpha(gh)} \overline{\chi_a(g)} \\ &= \frac{1}{|G|} \sum_{h \in G} \sum_{\alpha} f_\alpha(h) \langle R(h)f_\alpha, \chi_a \rangle = 0 \end{aligned}$$

4. FOURIER DECOMPOSITION

4.1. **Fourier Inversion.** As an immediate application of the previous theorem, for $f \in L^2(G)$ we have the following Fourier inversion formula

$$f(x) = \sum_{a,i,j} \langle f, \phi_{aij} \rangle \phi_{aij}(x), \quad \forall x \in G$$

We regard the following inner product as a Fourier coefficient:

$$\begin{aligned} \langle f, \phi_{aij} \rangle &= \frac{d_a^{1/2}}{|G|} \sum_{g \in G} f(g) \overline{\langle e_{ai}, \pi_a(g)e_{aj} \rangle} \\ &= \frac{d_a^{1/2}}{|G|} \sum_{g \in G} f(g) \langle \pi_a(g)e_{aj}, e_{ai} \rangle \\ &= d_a^{1/2} \langle \pi_a(f)e_{aj}, e_{ai} \rangle \end{aligned}$$

Hence we can re-write the Fourier inversion formula as

$$\begin{aligned} f(x) &= \sum_{a \in \widehat{G}} d_a \sum_{1 \leq i, j \leq d_a} (\pi_a(f)e_j, e_i) (e_i, \pi_a(x)e_j) = \sum_{a \in \widehat{G}} d_a \sum_{j=1}^{d_a} (\pi_a(f)e_j, \pi_a(x)e_j) \\ &= \sum_{a \in \widehat{G}} d_a \operatorname{trace}(\pi(x)^* \pi_a(f)) \\ &= \sum_{a \in \widehat{G}} d_a^2 \langle \pi_a(f), \pi_a(x) \rangle_{HS} \end{aligned}$$

4.2. **Parseval Identity.**

$$\begin{aligned} \|f\|_2^2 &= \sum_{a,i,j} |\langle f, \phi_{aij} \rangle|^2 = \sum_{a \in \widehat{G}} \sum_{1 \leq i, j \leq d_a} d_a |\langle \pi_a(f)e_j, e_i \rangle|^2 \\ &= \sum_{a \in \widehat{G}} d_a \operatorname{tr}(\pi_a(f)^* \pi_a(f)) \\ &= \sum_{a \in \widehat{G}} d_a \|\pi_a(f)\|^2 = \sum_{a \in \widehat{G}} d_a^2 \|\pi_a(f)\|_{HS}^2 \end{aligned}$$

4.3. **Example.** Assume G is a finite, abelian group. Then $\widehat{G} = G^*$ the group of one-dimensional characters. For $\chi \in G^*$, $d_\chi = 1$, therefore the above formulas yield

$$f(x) = \sum_{\chi \in G^*} \langle f, \chi \rangle \chi(x) = \sum \widehat{f}(\chi) \chi(x), \quad \|f\|_2^2 = \sum |\widehat{f}(\chi)|^2$$

5. HILBERT-SCHMIDT NORM

5.1. Assume V is a vector space of dimension d . On the vector space $End(V) = L(V, V)$ one has a canonical Hilbert structure given by

$$\langle A, B \rangle = \text{trace}(B^* A) = \sum_{i=1}^d \sum_{j=1}^d a_{ij} \bar{b}_{ij}$$

where (a_{ij}) and (b_{ij}) are the matrices associated to A and B , resp. with respect to an orthonormal basis $\{e_i\}$, $1 \leq i \leq d$. Note that $\langle A, A \rangle = \sum_{i,j} |a_{ij}|^2$, so this inner product is positive definite.

5.2. One also uses the normalization

$$\langle A, B \rangle_{HS} = \frac{1}{d} \langle A, B \rangle = \frac{1}{d} \sum_{i,j} a_{ij} \bar{b}_{ij}, \quad \|A\|_{HS}^2 = \frac{1}{d} \text{trace}(A^* A) = \frac{1}{d} \sum_{i,j} |a_{ij}|^2$$

such that $\|I\| = 1$, where I is the identity operator.