5. CHARACTER TABLE

1. Class functions

1.1. **Definition.** Let $L^2_{class}(G) \leq L^2(G)$ the linear subspace of functions that are invariant under conjugation,

$$L^2_{class}(G) = \{f \in L^2(G) : f(gxg^{-1}) = f(x), \quad \forall g, x \in G\}$$

Naturally $\chi_a \in L^2_{class}(G)$, $\forall a \in \widehat{G}$.

1.1.1. *Note.* $L^2_{class}(G)$ is **not** G-invariant (either left or right action).

1.2. **Theorem.**
1) $\{\chi_a : a \in \widehat{G}\}$ is an orthonormal basis for $L^2_{class}(G)$.
2) For $f \in L^2_{class}(G)$

$$f(x) = \sum_{a\widehat{G}} (\operatorname{tr} \pi_{a^*}(f))\, \chi_a(x), \quad \forall x \in G$$

1.2.1. *Proof.* 1) The only thing we need to prove is completeness of the family $\{\chi_a\}$ in L^2_{class}. Assume there exits $0 \neq f \in L^2_{class}(G)$ such that $f \perp \chi_a$, for all $a \in \widehat{G}$.

Let's first fix $a \in \widehat{G}$. Since f is a class function, $\pi_a(f) : V_a \to V_a$ is an intertwining operator, hence by Schur's lemma we have $\pi_a(f) = \lambda Id_{V_a}$. Taking the trace on both sides we obtain

$$\lambda \cdot d_a =< f, \chi_{a^*} >= 0$$

since f was chosen orthogonal on all χ_a. But then $\pi_a(f) = 0$. Since this holds for any $a \in \widehat{G}$, we obtain by the Fourier inversion formula

$$f(x) = \sum_{a\in\widehat{G}} d_a \operatorname{trace}(\pi_a(x)^* \pi_a(f)) = 0, \forall x \in G$$

contradiction.

2) The coefficients of f with respect to the χ_a basis are

$$\begin{aligned} < f, \chi_a > &= \frac{1}{|G|} \sum_{g\in G} f(g) \overline{\chi}_a(g) \\ &= \frac{1}{|G|} \sum_{g\in G} \sum_{j=1}^{d_a} f(g) < e_j, \pi_a(g) e_j >= \sum_{j=1}^{d_a} < e_j, \pi_a(f) e_j > \\ &= \overline{\operatorname{tr} \pi_a(f)} = \operatorname{tr} \pi_{a^*}(f) \end{aligned}$$

1.3. **Corollary.** Counting the dimension of L^2_{class} we obtain the identity

$$|\widehat{G}| = |Conj(G)|$$

1.4. **Theorem.** $\displaystyle\sum_{a\in\widehat{G}} d_a \chi_a(g) = \begin{cases} |G|, & g = e \\ 0, & g \neq e \end{cases}.$

1.4.1. *Proof.* The basis expansion of the delta function $\delta_e \in L^2_{class}$ yields

$$\delta_e(g) = \sum_{a\in\widehat{G}} < \delta_e, \chi_a > \chi_a(g) = \frac{1}{|G|} \sum_{a\in\widehat{G}} d_a \chi_a(g)$$

An alternative way to prove this identity is to recall the identity $\chi_{reg} = \sum_a d_a \chi_a$.

2. Character twists

2.1. Assume (π, V) is a representation of G and $\lambda : G \to \mathbb{C}^*$ is a group character. We define a new action $\pi \otimes \lambda$ of G on the same vector space V given by

$$\pi \otimes \lambda(g)(v) := \lambda(g)\pi(g)(v)$$

It is straightforward to check that the character of this representation is

$$\chi_{\pi\otimes\lambda}(g) = \lambda(g)\chi_\pi(g)$$

It is easy to establish the following

2.2. **Theorem.** If $\pi \in \widehat{G}$ is irreducible and $\lambda \in G^*$ is a group character, then $\pi \otimes \lambda$ is irreducible, $\pi \otimes \lambda \in \widehat{G}$.

This result gives a method for obtaining new irreducible representations from old ones, since in general (but not always!) $\pi \otimes \lambda \not\cong \pi$.

3. The Character Table

3.1. **Summary.**

$$\begin{cases} |\widehat{G}| = |Conj(G)| \\ d_a \mid |G| \\ \sum_{a\in\widehat{G}} d_a^2 = |G| \end{cases} \qquad \begin{cases} \sum_{g\in G} \chi_a(g) = |G| \; \delta_{a,\chi_0} \\ \sum_{a\in\widehat{G}} d_a\chi_a(g) = |G| \; \delta_{e,g} \\ \sum_{g\in G} \chi_a(g)\overline{\chi}_b(g) = |G| \; \delta_{a,b} \\ \chi_{a\otimes\lambda} = \lambda \cdot \chi_a \end{cases}$$

3.2. **Character table.**

G	${C_1}^{n_1}$	${C_2}^{n_2}$	$\dots$	${C_r}^{n_r}$
$d_1\ \pi_1$	$\chi_{\pi_1}(C_1)$	$\chi_{\pi_1}(C_2)$	$\dots$	$\chi_{\pi_1}(C_r)$
$d_2\ \pi_2$	$\chi_{\pi_2}(C_1)$	$\chi_{\pi_2}(C_2)$	$\dots$	$\chi_{\pi_2}(C_r)$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$d_r\ \pi_r$	$\chi_{\pi_r}(C_1)$	$\chi_{\pi_r}(C_2)$	$\dots$	$\chi_{\pi_r}(C_r)$

$n_i = |C_i|, \quad d_j = \dim \pi_j$

Here C_j are the conjugacy classes of G, n_j their corresponding cardinality; π_a are the irreducible representations, d_a the corresponding dimension. For finite groups, understanding the irreducible representations of G in many ways reduces to completing the character table: one starts with standard (obvious) representations of the group and aim to discover new ones using the orthogonality relations.

3.3. **More orthogonality (optional).** Let M the matrix with entries $m_{ij} = \chi_{\pi_i}(C_j)$. Let M^* with entries $(\tilde{M})_{ij} = |C_i|\overline{m}_{ji}$.

$$(M\tilde{M})_{ij} = \sum_k m_{ik}\overline{m}_{jk}|C_k| = \sum_k \chi_i(C_k)\overline{\chi}_j(C_k)|C_k| = \sum_{g\in G} \chi_i(g)\overline{\chi}_j(g) = |G|\delta_{ij}$$

Hence $(\tilde{M}M)_{ij} = |G|\delta_{ij}$ as well, that is

$$|G|\delta_{ij} = \sum_k (\tilde{M})_{ik}M_{kj} = \sum_k |C_i|\overline{m}_{ki}m_{kj} = |C_i| \sum_k \overline{\chi}_k(C_i)\chi_k(C_j)$$

In other words,

$$\sum_{a\in\widehat{G}} \chi_a(C_i)\chi_a(C_j) = \frac{|G|}{|C_i|}\;\delta_{ij}$$

3.4. **Example:** S_3**.**

S_3	e	$(12)_3$	$(123)_3$
$1\ \chi_0$	1	1	1
$1\ \epsilon$	1	-1	1
$2\ \chi_{st}$	2	0	-1

3.5. **Example:** S_4.

S_4	e_1	$(12)_6$	$(12)(34)_3$	$(123)_8$	$(1234)_6$
${}_1\chi_0$	1	1	1	1	1
${}_1\epsilon$	1	-1	1	1	-1
${}_3\pi_{st}$	3	1	-1	0	-1
${}_3\pi_{st} \otimes \epsilon$	3	-1	-1	0	1
${}_2\pi_W$	2	0	2	-1	0

Strategy: we start out with $\chi_0, \epsilon, \pi_{st}, \pi_{st} \otimes \epsilon$. $\sum_a d_a^2 = 24 \Rightarrow$ there is one more irreducible representation in dimension 2. We deduce the character from the relation: $\sum_a d_a \chi_a(g) = |G|\ \delta_{e,g}$. To obtain an explicit construction of a representation with character χ_W, we notice that

$$\pi_W((12)(34)) = Id_W$$

This follows from the fact that $\pi((12)(34))$ is unitary, self-adjoint, of trace 2. Hence

$$K = \{e, (12)(34), (13)(24), (14)(23)\} \subset ker(\pi_W)$$

Hence $\pi_W : G/K \to GL(W)$. But $G/K \simeq S_3$ and W is a unique representation for G/K, so W is just the standard representation of S_3:

$$S_4 \to S_4/K \simeq S_3 \to GL(W_{2,st})$$

3.6. **Example:** A_4.

A_4	e_1	$(12)(34)_3$	$(123)_4$	$(132)_4$
${}_1\chi_0$	1	1	1	1
${}_1\omega$	1	1	ζ	ζ^2
${}_1\omega^2$	1	1	ζ^2	ζ
${}_3\chi_\rho$	3	-1	0	0

Strategy: first one determines the characters. Here $A_4/K \simeq \mathbb{Z}/3\mathbb{Z}$, hence A_4 inherits the characters ω, ω^2 of the cyclic group. By dimension counting: there is one more representation ρ in dimension 3. By comparing the characters, we see that $\rho = \mathrm{Res}_{A_4}^{S_4} \pi_{st}$.

Note: let C a conjugacy class of S_4. Then either $C \subset A_4$ or $C \cap A_4 = \emptyset$. Hence the conjugacy classes of S_4 may break up into disjoint unions of conjugacy classes of A_4. This is the case of $C_{(123)} = C' \cap C''$, with $C' = C((123)$ and $C'' = C((132))$.

4. Tensor Products

4.1. **Definition.** For V, W vector spaces over $\mathbb{C}$. By definition, $V \otimes W$ is the unque vector spaces with the property that it filters all bilinear maps $V \times W \to Z$. Concretely, $V \otimes W$ is the vector space of dimension $\dim(V \otimes W) = (\dim V) \cdot (\dim W)$ spanned by linearly independent vectors $e_i \otimes f_j$, where e_i is a basis on V and f_j is a basis on W. Concretely, $V \otimes W = (V \times W)/\mathcal{U}$, where U is the $\mathbb{C}$-linear span of the vectors of type

$$(cv_1, v_2) - (v_1, cv_2); \quad (v_1 + w_1, v_2) - (v_1, v_2) - (w_1, v_2); \quad (v_1, v_2 + w_2) - (v_1, v_2) - (v_1, w_2)$$

4.2. **Functorial Property.** $V \otimes W$ is the unique vector space (up to isomorphism) with the following property:

given a vector space Z and any bilinear map $B : V \times W \to Z$, $\exists\,!\ \varphi_B \in L(V \otimes Z)$ such that

$$B(v, w) = \varphi_B(v \otimes w), \quad \forall v \in V, w \in W$$

4.3. **Canonical isomorphism.** We have a vector space isomorphism $T : V^* \otimes W \simeq \mathcal{L}(V, W)$ defined on pure tensors by $T(v^* \otimes w)(x) = v^*(x)w$ and extended to general tensors (not necessarily pure) by linearity. For example, if the operator $A \in \mathcal{L}(V, W)$ has the matrix (a_{ij}) with respect to some $\{e_i\}$ basis on V and $\{f_j\}$ basis on W, then $A = T(\sum a_{ij}\ e_j \otimes f_i^*)$, where f_i^* is the dual basis. We will simply write $A = \sum_{i,j} a_{ij} e_j \otimes f_i^*$ without writing T.

Note: the elements of the dual basis are $f_i \in V^*$ given by $f_i^*(f_j) = \delta_{ij}$.

4.4. **Hilbert structure.** The tensor product of two vector spaces with inner products has a canonical inner product. If e_i and f_j are ON basis on V, W resp., then there is a unique Hilbert structure on $V \otimes W$ such that $< e_i \otimes f_j, e_\alpha \otimes f_\beta >= \delta_{i\alpha}\delta_{j\beta}$.

In particular the inner product on $V^* \otimes V \simeq End(V)$ is given by the Hilbert-Schmidt inner product

$$< A, B >= \operatorname{tr}(B^* A) = \sum_{ij} a_{ij}\bar{b}_{ij} = \sum_i < Ae_i, Be_i >$$

We normalize this inner product by $< A, B >_{HS}= \frac{1}{\dim V} < A, B >$ for convenience, so that $\|1_V\|_{HS} = 1$.

4.5. **External Tensor Product.** If (π, V) is a representation of G and (σ, W) a representation of H, then $V \otimes W$ carries a representation of $G \times H$, call its $\pi \boxtimes \sigma$, given on pure tensors $v \otimes w$ by

$$\pi \boxtimes \sigma(g, h)(v \otimes w) = \pi(g)v \otimes \sigma(h)w$$

and extended to general tensors (non necessarily pure) by linearity:

$$\pi \boxtimes \sigma(g, h)(\sum_{i,j} a_{ij} v_i \otimes w_j) = \sum_{i,j} a_{ij}\pi(g)(v_i) \otimes \sigma(h)w_j, \quad a_{ij} \in \mathbb{C}$$

4.6. **Theorem.**

a) $\chi_{\pi\boxtimes\sigma}(g, h) = \chi_\pi(g)\chi_\sigma(h)$.

b) If $\pi \in G$ is irreducible for G and σ is irreducible for H, then $\pi \boxtimes \sigma$ is irreducible for $G \times H$. Loosely speaking, this means $\widehat{G} \otimes \widehat{H} \subset \widehat{G \times H}$.

c) The converse statement is true: all irreducible representations of $G \times H$ occur this way. In other words, we have the equality

$$\widehat{G} \otimes \widehat{H} = \widehat{G \times H}$$

4.7. **Internal Tensor product.** Assume (π, V) and (σ, W) are two representations of the same group G. Then $\pi \otimes \sigma$ is the representation of G on $V \otimes W$ given by $(\pi \otimes \sigma)(g)(v \otimes w) = \pi(g)v \otimes \pi(g)w$.

4.8. **Theorem.** $\chi_{\pi\otimes\sigma} = \chi_\pi\chi_\sigma$.

4.8.1. *Note.* In general $\pi \otimes \sigma$ is no longer an irreducible representation of G-see below. (But $\pi \boxtimes \sigma$ is irreducible for $G \times G$.)

4.9. $\wedge^2$ **and** sym^2**.** When $V = W$, every element of $V \otimes V$ can be written uniquely as a sum of a symmetric and an anti-symmetric tensor. In other words,

$$V \otimes V = \wedge^2 V \oplus \operatorname{sym}^2 V$$

If (π, V) is a representation of G, then $\wedge^2 V$ and $\operatorname{sym}^2 V$ are G-invariant subspaces of $V \otimes V$, and hence they are representations of G. We denote the action of G on them by $\wedge^2\pi$ and $\operatorname{sym}^2 \pi$ resp.

4.9.1. *Trace.*

$$\chi_{\wedge^2\pi}(g) = \frac{1}{2}(\chi_\pi(g)^2 - \chi_\pi(g^2)), \quad \chi_{\operatorname{sym}^2\pi}(g) = \frac{1}{2}(\chi_\pi(g)^2 + \chi_\pi(g^2))$$

4.10. **Commuting representations.** Assume π, ρ are two representations of G on the same space V. If $\pi(g)\rho(h) = \rho(h)\pi(g)$ for all $g, h \in G$, then one can define the action $\pi \times \rho$ of $G \times G$ on the space V given by

$$\pi \times \rho(g)v = \pi(g)\rho(g)v$$

This defines a representation of $G \times G$ on V.

4.11. **Peter-Weyl revisited.** We have a $G \times G$ representation on $L^2(G)$, namely $L \times R$. The map $\Phi_a : V_a^* \otimes V_a \to L^2(G)$ given by $\Phi_a(u^* \otimes v)(x) = u^*(\pi_a(g)v)$ is $G \times G$-equivariant and is norm preserving, that is

$$\|\Phi_a(A)\|_{L^2(G)} = \|A\|_{HS}, \quad \forall A \in V_a^* \otimes V_a \simeq End(V_a)$$

Moreover, the images of different Φ_a are orthogonal inside $L^2(G)$ and this gives an isomorphism of $G \times G$ representations

$$L^2(G) \simeq \bigoplus_{a\in\widehat{G}} V_a^* \otimes V_a$$