6. THE EXPONENTIAL MAP ON $GL(n,\mathbb{C})$

1. Hilbert-Schmidt norm

1.1. **Definition.** $gl(n,\mathbb{C})$ the complex vector space of $n\times n$ matrices with complex entries. The Hilbert-Schmidt inner product on this space is given by

$$< A, B >= \mathrm{trace}(B^*A) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,\bar{b}_{ij}$$

The associated norm is $\|A\| = \mathrm{trace}(A^*A)^{1/2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$.

1.2. **Definition.** A matrix $A\in gl(n,\mathbb{C})$ is Hermitian if $A^*=A$ or $a_{ji}=\overline{a}_{ij}$. A matrix $A\in gl(n,\mathbb{C})$ is positive semi-definite (notation: $A\ge 0$) iff $< Ax, x >= x^tA\overline{x}\ge 0$, $\forall x\in\mathbb{C}^n$.

1.3. **Lemma.** If $A=A^*\ge 0$ and $B=B^*\ge 0$, then $\mathrm{tr}(AB)\le \mathrm{tr}(A)\,\mathrm{tr}(B)$.

1.3.1. *Proof.* Let $\{e_k\}$ an orthonormal eigenbasis for A, $Ae_k=\lambda_k e_k$, $1\le k\le n$. Then

$$\mathrm{tr}(AB)=\mathrm{tr}(BA)=\sum_k < BAe_k, e_k >=\sum_k \lambda_k < Be_k, e_k >\le (\sum_k \lambda_k)(\sum_k < Be_k, e_k >)$$

by positivity.

1.4. **Proposition.** For $A,B\in gl(n,\mathbb{C})$, $\|AB\|\le\|A\|\cdot\|B\|$.

1.4.1. *Proof.* $\|AB\|^2=\mathrm{tr}(B^*A^*AB)=\mathrm{tr}(A^*ABB^*)\le \mathrm{tr}(A^*A)\,\mathrm{tr}(BB^*)$ by the lemma.

2. Exponential map

2.1. **Theorem.** For $X\in gl(n,\mathbb{C})$, the series $\sum_{k=0}^{\infty}\frac{1}{k!}X^k$ converges uniformly on compact subsets of $gl(n,\mathbb{C})$.

2.1.1. *Proof.* The series of absolute values is $\sum_k \|X^k\|/k! \le \sum_k \|X\|^k/k! = e^{\|X\|}$.

2.2. **Definition.** We denote $\exp(X)=e^X=\sum_{k=0}^{\infty}\frac{1}{k!}X^k$.

2.3. **Properties of the exponential map.**

2.3.1. $\exp(sX)\exp(tX)=\exp((s+t)X)$, $\forall X\in gl(n,\mathbb{C})$ and $s,t\in\mathbb{R}$.

This follows from interchanging the order of summation, allowed for absolutely convergent series.

2.3.2. $\exp(X)\in GL(n,\mathbb{C})$, $\exp(X)^{-1}=\exp(-X)$.

2.3.3. $\det(\exp X)=e^{\mathrm{trace}\,X}$.

This follows by writing the matrix X in the Jordan canonical form.

2.3.4. $\exp(gXg^{-1})=g\exp(X)g^{-1}$, for $g\in GL(n,\mathbb{C})$ and $X\in gl(n,\mathbb{C})$.

This follows from $(gXg^{-1})^n=gX^ng^{-1}$.

2.4. For $\|X\|<1$, the series $\log(1+X)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}X^n$ converges absolutely and $\exp(\log(1+X))=1+X$. Thus $\exp^{-1}g=\log(g)$ for $\|g-1\|<1$. In particular, we have a diffeomorphism

$$\exp : U\simeq V$$

where $0\in U\subset gl(n,\mathbb{C})$ is a small open neighborhood of 0 in $gl(n,\mathbb{C})$ and $e\in V\subset GL(n,\mathbb{C})$ is a small open neighborhood of e in $GL(n,\mathbb{C})$.

2.5. **Charts.** The exponential map provides natural charts for $GL(n, \mathbb{C})$. For an arbitrary $g \in G$, the set $V_g := gV$ is an open neighborhood of g and

$$V_g \xrightarrow{g^{-1}} V \xrightarrow{\log} U$$

gives a chart near g. In other words, $V_g := \{g \exp(X) : X \in U\}$ is a neighborhood of $g \in G$, parametrized by an open set of the origin $U \subset \mathbb{C}^{n^2}$.

3. Campbell-Hausdorff

3.1. **Definition.** For $\|X\|, \|Y\| < \frac{1}{10}$ (small) we define $M(X,Y) := \exp^{-1}(\exp(X)\exp(Y))$. defined for X and Y small ($\|X\|, \|Y\| < \frac{1}{10}$). Clearly $M(X,0) = X$ and $M(0,Y) = Y$.

3.2. **Theorem.** $\dfrac{\partial^2}{\partial s \partial t} M(sX, tY)|_{s=t=0} = \dfrac{1}{2}[X,Y]$.

3.3. **Taylor Expansion.** This means: $M(X,Y) = X + Y + \frac{1}{2}[X,Y] +$ [higher order terms] or more specifically

$$M(sX, tY) = sX + tY + \frac{st}{2}[X,Y] + o(s^2 + t^2), \quad \text{as } (s,t) \to (0,0)$$

4. One parameter subgroups

4.1. **Definition.** The map $\gamma_X : \mathbb{R} \to GL(n, \mathbb{C})$ defined by $\gamma_X(t) := \exp(tX)$ has the following properties:

$$\gamma_X(t) \text{ is continuous in } t; \quad \gamma_X(s+t) = \gamma_X(s)\gamma_X(t), \quad \forall s,t \in \mathbb{R}$$

We call such a map a one-parameter subgroup of $GL(n, \mathbb{C})$. We also say that X is the generator of the one-parameters subgroup X.

4.2. **Theorem.** All one-parameter subgroups are of this form [homework].

4.3. We recover the generator X from the one-parameter subgroup $\gamma(t)$ by

$$X = \frac{d}{dt}\gamma(t)|_{t=0}$$