7. LIE GROUPS AND LIE ALGEBRAS

1. Lie algebras

1.1. **Definition.** Here $F = \mathbb{R}$ or $\mathbb{C}$. A Lie algebra over F is a pair $(\mathfrak{g}, [\cdot,\cdot])$, where $\mathfrak{g}$ is a vector space over F and

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is an F-bilinear map satisfying the following properties

$$[X, Y] = -[Y, X]$$
$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

The latter is the Jacobi identity.

1.1.1. *Note.* $[x, x] = 0$, $\forall x \in \mathfrak{g}$.

1.2. **Example.** If V is a vector space over F, $gl_F(V) = (\mathcal{L}(V), [\cdot,\cdot]_{op})$ is a Lie algebra over F, where the (operator) Lie bracket is given by

$$[A, B]_{op} = AB - BA$$

If $V = F^n$ then we will use the notation $gl_n(F)$ or $gl(n, F)$.

1.3. **Homomorphism of Lie algebras.** An F-linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras over F is called a homomorphism of Lie algebras if it preserves the bracket in the following manner

$$[\phi(x), \phi(y)] = \phi([x, y]), \quad \forall x, y \in \mathfrak{g}_1$$

Note that the bracket on the left-hand side is taken in $\mathfrak{g}_2$, while on the right-hand side is taken in $\mathfrak{g}$.

1.4. **Abelian Lie algebras.** In general, the Lie algebra structure on a given vector space is not unique. For example, we can endow *any* vector space V (over F) with the trivial Lie algebra structure given by

$$[x, y]_{ab} = 0, \quad \forall x, y \in V$$

From now on, we will use the words *abelian Lie algebra* or *commutative Lie algebra* to refer to a Lie algebra with a trivial Lie bracket.

1.4.1. *Note.* From now on we will refer to the Lie algebra as $\mathfrak{g}$ instead of $(\mathfrak{g}, [\cdot,\cdot])$, without specifying the bracket (assuming it is understood).

1.5. **One-dimensional Lie algebras.** Since $[x, x] = 0$, it means that a Lie algebra of dimension one is necessarily abelian. Thus to obtain non-trivial examples of Lie algebra one has to look in dimension greater or equal to two.

1.6. **Adjoint action.** Assume $\mathfrak{g}$ is a Lie algebra over F and $gl(\mathfrak{g})$ is the Lie algebra of linear operators on $\mathfrak{g}$. Consider the following map

$$\mathrm{ad} : \mathfrak{g} \to gl(\mathfrak{g}), \quad \mathrm{ad}(x)y = [x, y], \quad \forall x, y \in \mathfrak{g}$$

We call the map $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ the adjoint action of the element x on $\mathfrak{g}$. It is straightforward to check that <u>ad</u> is a homomorphism of Lie algebra:

$$\mathrm{ad}(x)([y, z]) = [\mathrm{ad}(x)y, \mathrm{ad}(x)z]$$

In fact, this is equivalent to the Jacobi identity.

2. The Adjoint Representation

2.1. **Definition.** In this section $G = GL(n, \mathbb{C})$ and $\mathfrak{g} = gl(n, \mathbb{C})$. For $g \in G$, we define the map

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}(g)(X) = gXg^{-1}, \quad \forall X \in \mathfrak{g}$$

This gives a representation of G on $\mathfrak{g}$, in that

$$\mathrm{Ad}(g)\,\mathrm{Ad}(h) = \mathrm{Ad}(gh)$$

In other words, the map

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$

is a group homomorphism. We call this representation the *adjoint representation.*

2.1.1. *Note.* $\dim_{\mathbb{C}} \mathfrak{g} = n^2$, so a matrix representation of $\mathrm{Ad}(g)$ would require $n^2 \times n^2$ matrix. Already for $n = 3$ this is a matrix with 81 entries. It might be easier at this point to appreciate the point of view that a representation assigns to a group element a linear operator rather than a matrix.

2.2. **Conjugation.** $ge^X g^{-1} = e^{\mathrm{Ad}(g)X}$, for $g \in G$ and $X \in \mathfrak{g}$.

This follows from the identity $gX^n g^{-1} = (gXg^{-1})^n$.

2.3. **Theorem.** The following diagram is commutative

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & gl(\mathfrak{g}) \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathrm{Ad}} & GL(\mathfrak{g}) \end{array}$$

That is, we have an identity of linear operators

$$\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\,X} : \mathfrak{g} \to \mathfrak{g}, \quad \forall X \in \mathfrak{g}$$

2.3.1. *Proof.* $\mathrm{Ad}(\exp tX)$ is a (continuous) one-parameter subgroup of $GL(\mathfrak{g})$ (check). Therefore there exists $A \in gl(\mathfrak{g})$ such that $\mathrm{Ad}(\exp tX) = \exp(tA)$. The generator of this one-parameter group is given by

$$A = \frac{d}{dt}\,\mathrm{Ad}(\exp tX)|_{t=0}$$

This means that for $Y \in \mathfrak{g}$, $A(Y)$ is given by

$$A(Y) = \frac{d}{dt}\,\mathrm{Ad}(\exp(tX))Y|_{t=0} = \frac{d}{dt}\, e^{tX} Y e^{-tX}|_{t=0} = XY - YX = [X, Y] = \mathrm{ad}(X)Y$$

by Leibniz rule.

2.3.2. *Equivalent formulation.* In other words, for $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned} e^X \cdot Y \cdot e^{-X} &= e^{\mathrm{ad}\,X}(Y) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathrm{ad}(X)^n(Y) \\ &= Y + \frac{1}{1!}[X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \dots \end{aligned}$$

3. Lie subgroups of $GL(n, \mathbb{C})$

3.1. **Definition.** A Lie subgroup of $GL(n, \mathbb{C})$ is a *closed subgroup* of $GL(n, \mathbb{C})$. From now on, a Lie group will be either a Lie subgroup of $GL(n, \mathbb{C})$, or a group isomorphic to it. Note that a closed subgroup of a Lie group is automatically a Lie group.

3.2. **Topology.** Since we can think of $GL(n, \mathbb{C})$ as an open subset of $\mathbb{C}^{n^2}$, its topology is the one inherited from $\mathbb{C}^{n^2}$. In other words, the sequence $\{g_k\}$ with $g_k \in GL(n, \mathbb{C})$ converges to $g \in GL(n, \mathbb{C})$ if and only if $(g_k)_{ij} \to g_{ij}$, for all $1 \le i, j \le n$, where $g(ij)$ is the ij-entry of the matrix g.

A subgroup $H \subset GL(n, \mathbb{C})$ is *closed* it it has the following property:

$$\text{if } g_n \to g \in GL(n, \mathbb{C}) \text{ and } g_n \in H, \text{ then } g \in H \text{ as well}$$

3.3. **Examples.** $\mathbb{R}_{>0}$ is a closed subgroup of $GL(1, \mathbb{C})$, hence a Lie group.

The group isomorphism $\exp : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$ identifies $\mathbb{R}$ with $\mathbb{R}_{>0}$. This makes $\mathbb{R}$ itself a Lie group.

4. The Lie algebra of a Lie subgroup

4.1. **Definition.** If $G \subseteq GL(n,\mathbb{C})$ is a closed subgroup, we define its Lie algebra to be the set

$$\mathfrak{g} := Lie(G) = \{X \in gl(n,\mathbb{C}) : \exp(tX) \in G, \forall t \in \mathbb{R}\}$$

4.2. **Theorem A.** $\mathfrak{g}$ is a Lie subalgebra of $gl(n,\mathbb{C})$ over $\mathbb{R}$, meaning:

- $tX \in \mathfrak{g}$, $\forall t \in \mathbb{R}$
- $X + Y \in \mathfrak{g}$
- $[X,Y] \in \mathfrak{g}$

whenever $X, Y \in \mathfrak{g}$.

4.2.1. *Proof.* The first relation follows from the definition.

For the second part, we need the following

4.2.2. *Lemma.* For $X, Y \in gl(n,\mathbb{C})$,

$$\lim_{n\to+\infty} (\exp(X/n)\exp(Y/n))^n = \exp(X+Y)$$

To prove the lemma, note that the left-hand side is

$$(\exp M(X/n, Y/n))^n = \exp(nM(X/n,Y/n)) = \exp(n(X/n + Y/n + O(1/n^2)) = \exp(X + Y + O(1/n))$$

To see how to use this formula to prove the second part of the Theorem, note that for $X, Y \in \mathfrak{g}$,

$$\exp(t(X+Y) = \lim_n \big(\exp(tX/n)\exp(tY/n)\big)^n$$

The terms of the sequence of right-hand side belong to G. But G is closed, so the limit belongs to G as well, that is $\exp(t(X+Y)) \in G$. Since this is true for all $t \in \mathbb{R}$, it means that $X + Y \in \mathfrak{g}$.

For the last part of the Theorem, we use [homework]

$$\lim_{n\to\infty} (\exp(X/n)\exp(Y/n)\exp(-X/n)\exp(-Y/n))^{n^2} = \exp([X,Y])$$

4.3. **The Adjoint Representation.** If $g \in G$ and $X \in \mathfrak{g}$, then $\mathrm{Ad}(g)X \in G$. To see this,

$$\exp(t\,\mathrm{Ad}(g)X) = g\exp(tX)g^{-1} \in G, \quad \forall t \in \mathbb{R}$$

Hence $\mathrm{Ad}(g)X \in \mathfrak{g}$. We thus obtain the adjoint representation $\mathrm{Ad} : G \to gl(\mathfrak{g})$ and a commutative diagram as in the case of $GL(n,\mathbb{C})$.

4.4. **Note.** Let $\mathfrak{g}$ the Lie algebra of a Lie group G. Then the bracket operation does not depend on the particular embedding $G \subset GL(n,\mathbb{C})$. On the other hand, the matrix product XY does. This is the reason why it is necessary to think of $[,]$ as *the* natural operation on $\mathfrak{g}$.

5. Cartan's Theorem

5.1. **Theorem B.** If $G \subset GL(n,\mathbb{C})$ is a closed subgroup, then there exists neighborhoods $0 \in U \subset gl(n,\mathbb{C})$ and $e \in V \in GL(n,\mathbb{C})$ such that

$$\exp : U \simeq V, \quad \exp(U \cap \mathfrak{g}) = V \cap G$$

5.1.1. *Note.* This shows that closed Lie subgroups of $GL(n,\mathbb{C})$ are smooth sumbmanifolds of $GL(n,\mathbb{C})$.

6. Discrete subgroups

6.1. **Definition.** A subgroup $\Gamma \subset G$ is *discrete* if for any $\gamma \in \Gamma$ there exists an open neighborhood $\gamma \in V \subset G$ such that $V \cap \Gamma = \{\gamma\}$. A discrete subgroup is automatically closed, hence it is a Lie subgroup. Its Lie algebra is trivial, $Lie(\Gamma) = \{0\}$.

6.2. **Examples.** $\mathbb{Z} \subset \mathbb{R}$ and $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$.

7. Example: $SL(2,\mathbb{R})$

This is clearly a closed subgroup of $GL(2,\mathbb{R})$, hence a Lie subgroup.

7.1. **Lie algebra.** Let $sl(2,\mathbb{R}) := Lie(SL(2,\mathbb{R}))$. By definition, $X \in sl(2,\mathbb{R})$ iff $\det\exp(tX) = 1$, for all $t \in \mathbb{R}$. Since $\det\exp(tX) = e^{t\,\mathrm{trace}(X)}$, we have

$$sl(2,\mathbb{R}) = \{X : \mathrm{trace}\,X = 0\}$$

7.2. **Generators.** The matrices

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

form a basis for $sl(2,\mathbb{R})$ as a vector space over $\mathbb{R}$. They satisfy the relations

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H$$

We say that $\{H, E_+, E_-\}$ form an $sl(2)$-triple.

7.2.1. *Note.* $H \cdot E_+ \notin sl(2,\mathbb{R})$ (matrix multiplication) but $[H, E_+] \in sl(2,\mathbb{R})$.

7.3. $sl(2)$ **triples.** To show that another (real) Lie algebra $\mathfrak{h}$ is isomorphic to $sl(2,\mathbb{R})$ it is enough to find a linear basis x, y_+, y_- of $\mathfrak{h}$ such that

$$[x, y_\pm] = \pm 2y_\pm, \quad [y_+, y_-] = x$$

7.4. **Exponential map.** $\exp : sl(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is not surjective, although $SL(2,\mathbb{R})$ is connected. To see this, take $g = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Then $g \notin \exp(sl(2,\mathbb{R}))$. For assume that was the case, $g = \exp X$, $\operatorname{trace} X = 0$. We know that X is not diagonalizable (since g is not) which means that X has a double eigenvalue. Since $\operatorname{trace} X = 0$, the double eigenvalue $= 0$. But this implies that $g = e^X$ has eigenvalue 1, contradiction.

7.5. **Inverse in** $SL(2)$**.** Useful formula: for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$, $g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

8. Example: $O(2), SO(2)$

8.1. **Definition.** $O(2) = \{g \in SL(2,\mathbb{R}) : g^t g = I\}, \quad SO(2) = \{g \in O(2) : \det g = 1\}$.

8.2. **Lie algebra.** $o(2) := Lie(O(2)) = \{X \in gl(2,\mathbb{R}) : \exp(tX)\exp(tX^\tau) = I_2, \forall t \in \mathbb{R})$. By differentiating this identity at $t = 0$

$$X^t + X = \frac{d}{dt}\ \exp(tX^t)\exp(tX)|_{t=0} = 0$$

Hence $o(2) = \{X \in gl(2,\mathbb{R}) : X^t = -X\}$. Note that an antisymmetric matrix has zero on the diagonal, and hence zero trace. This means that $so(2) = o(2)$. Note that $o(2) = so(2) = \{\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}, t \in \mathbb{R}\}$ is generated by $H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then

$$SO(2) = \exp(tH) = \{\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, t \in \mathbb{R}\} \simeq \mathbb{S}^1$$

and $[O(2) : SO(2)] = 2$. Then $O(2)$ is the disjoint union $O(2) = SO(2) \bigcup gSO(2)$, with $g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

9. Example: $SU(2)$

9.1. **Definition.** $SU(2) = \{g \in GL(2,\mathbb{C}) : \det g = 1,\ \overline{g}^t g = I_2\}$. Note that

$$g^t = g^{-1} \Rightarrow \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence $SU(2) = \{g = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix} : |a|^2 + |b|^2 = 1\}$. Topologically $SU(2) \simeq S^3$.

9.2. **Lie algebra.** $su(2) := \{X \in gl(2,\mathbb{C}) : \overline{X}^t + X = 0\} = \{\begin{bmatrix} ia & \beta \\ -\overline{\beta} & -ia \end{bmatrix} : a \in \mathbb{R}, \beta \in \mathbb{C}\}$.

10. Connected component

10.1. **Definition.** $G^0 :=$ connected component of $e \in G$. In other words, G^0 is the set of points g with the property that there exists an compact interval $[a, b] \subset \mathbb{R}$ and continuous map $c : [a, b] \to G$ such that $c(a) = e$ and $c(b) = g$. We say that $c(t)$ is a *path* joining e and g.

10.2. **Theorem.**

a) G^0 is both open and closed in G.

b) Then $G^0 \lhd G$ is a normal subgroup of G.

c) Let $e \in V \subset G^0$ an open, symmetric, connected subset of G^0. Then $G^0 =< V >$, i.e. the set V generates the group G^0.

d) $Lie(G^0) = Lie(G)$.

10.2.1. *Proof.* a) Assume $x, y \in G^0$. Take c_1 a path in G joining e and x; c_2 a path joining e and y. Then the concatenation of the translated paths $c_1(t)^{-1}x$ and $x^{-1}c_2(t)$ is a path joining e and $x^{-1}y$. Therefore $x^{-1}y \in G^0$, which shows that G^0 is a subgroup of G. Being the connected component of e, G^0 is at once open and closed relative to G (this is a general fact).
b) Assume $g \in G$ and $x \in G^0$. If $c(t)$ is a path joining e and x, then $g \cdot c(t) \cdot g^{-1}$ is a (continuous) path joining e and gxg^{-1}, hence $gxg^{-1} \in G^0$.
c) The statement of part c) is

$$G^0 = \cup_{n=0}^{\infty} V^n, \quad \text{where} \quad V^n := \{g = x_1 \cdot \dots \cdot x_n : x_1, \dots, x_n \in V\}$$

But this is true since the right-hand side is an open subgroup of G^0 and hence also closed relative to G^0.
d) G^0 being a closed subgroup of G, it is a Lie subgroup. Hence it makes sense to talk about its Lie algebra. The identity of the two Lie algebras is immediate once we notice that a continuous path (such as a one-parameter subgroup) passing through e stays in G^0.

10.2.2. *Note.* Assume G is a Lie group and $H \subset G$ is an open subgroup. Then H is also closed . To see this, write G as a disjoint union of left H-cosets: $G = \cup_{i \in I} g_i H$, where say $g_0 = e$. Then $G - H = \cup_{i \neq 0} g_i H$ is a union of open sets, hence open. But this means that H itself is closed.

10.3. **Theorem C.** Assume G and H are two Lie subgroups of $GL(n, \mathbb{C})$ with the same Lie algebra. Then $G^0 = H^0$.

10.3.1. *Proof.* Assume $\mathfrak{g} = \mathfrak{h}$, where $G, H \subset GL(n, \mathbb{C})$ are closed subgroups. Let U a small neighborhood of $0 \in gl(n, \mathbb{C})$ such that $\exp : U \simeq V$, with V a neighborhood of $I \in GL(n, \mathbb{C})$. By Theorem B, $\exp(U \cap \mathfrak{g})$ is an open connected neighborhood of both $e \in G$ and $e \in H$. Two groups generated by the same elements are identical:

$$G^0 =< \exp(U \cap \mathfrak{g}) >= H^0$$

10.4. **Examples.** For $n \geq 2$, the Lie groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $SU(n)$ and $SO(n)$ are connected while $O(n)$ is not connected. $SO(n) = O(n)^0$ and $[O(n) : SO(n)] = 2$.