8. COMPACT LIE GROUPS AND REPRESENTATIONS

1. ABELIAN LIE GROUPS

1.1. Theorem. Assume $G$ is a Lie group and $\mathfrak{g}$ its Lie algebra. Then

$G^0$ is abelian if $\mathfrak{g}$ is abelian.

1.1.1. Proof. "$\Rightarrow".$ Let $0 \in U \subset \mathfrak{g}$ and $e \in V \subset G$ small (symmetric) neighborhoods of $0$ in $\mathfrak{g}$ and resp. of $e$ in $G$ such that $\exp : U \simeq V$. Pick $x, y \in V$ and let $x = \exp X, y = \exp Y$, with $X, Y \in U$. Then

$$xy = \exp X \exp Y = \exp M(X, Y) = \exp(X + Y) = \exp(Y) \exp(X) = yx$$

Since $V$ generates $G^0$, it follows that $G^0$ is abelian.

"$\Rightarrow".$ Let $g \in G^0$ and $Y \in \mathfrak{g}$. Since $\exp(tY) \in G^0$, $g \exp(tY)g^{-1} = \exp(tY)$. Differentiating this identity at $t = 0$ we obtain $\text{Ad}(g)Y = Y$.

Let now $X, Y \in \mathfrak{g}$. Since $\exp(tX) \in G^0$, $\text{Ad}(\exp(tX))Y = Y$. Differentiating the identity at $t = 0$ we obtain $\text{ad}(X)Y = 0$, so $\mathfrak{g}$ is abelian.

1.2. Example. The Lie algebra of $O(2)$ is abelian (one-dimensional), yet $O(2)$ is not abelian:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

However its connected component $SO(2)$ is connected.

2. EXPLICIT EXAMPLES OF ABELIAN LIE GROUPS

2.1. $G_1 = \mathbb{R}_{>0}$. The group operation is $x \cdot y = xy$. Then $G_1 = G^0$, where $G = GL(1, \mathbb{R})$. Hence $\text{Lie}(G_1) = \text{Lie}(G^0) = \mathfrak{gl}(1, \mathbb{R}) \simeq \mathbb{R}$ and $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is given by $\exp(t) = e^t$.

$$\mathfrak{gl}(1, \mathbb{R}) \xrightarrow{\exp} G_1 \subset GL(1, \mathbb{R})$$

2.2. $G_2 = (\mathbb{R}, +)$. The group structure is additive $x \cdot y = x + y$. We realize $G_2$ as a matrix Lie group via the group isomorphism $\phi : G_2 \to G_1, \phi(x) = e^x$. Hence $\text{Lie}(G_2) = \text{Lie}(G_1) = \mathbb{R}$ and the exponential map pulls back:

$$\exp_{G_2} : \mathbb{R} \to G_2, \quad \phi \circ \exp_{G_2} = \exp_{G_1}$$

So it simply given by $\exp_{G_2} : \mathbb{R} \to \mathbb{R}, \exp_{G_2}(t) = t$.

2.3. In general, $\text{Lie}(G \times H) = \mathfrak{g} \times \mathfrak{h}$, and $\exp_{G \times H}(X, Y) = (\exp_G(X), \exp_H(Y))$. In particular, for any finite dimensional real vector space $V$, $G_3 = (V, +)$ is a Lie group with abelian Lie algebra $\text{Lie}((V, +)) = (V, [, ]_{ab})$ and exponential map $\exp_V = i1_V : V \to V$. Such an abelian Lie group is called a vector group.

2.4. Similarly $G_4 := \mathbb{C}^\times \subset GL(1, \mathbb{C})$ is a Lie group with Lie algebra $\text{Lie}(G_4) = \mathbb{C}$ and exponential map $\exp_{G_4} : \mathbb{C} \to \mathbb{C}^\times, \exp_{G_4}(z) = e^z$.

2.5. $G_5 = S^1 := \{e^{i\theta} : \theta \in \mathbb{R}\} \subset \mathbb{C}^\times$. Then $\text{Lie}(S^1) = i\mathbb{R} \subset \mathbb{C} = \text{Lie}(\mathbb{C}^\times)$ and $\exp_{S^1} : i\mathbb{R} \to S^1$ is the usual exponential. Equivalently, we can think of $\text{Lie}(S^1) = \mathbb{R}$ with $\exp_{S^1}(t) = e^{it}$.

There is an alternative realization of $S^1$ as a matrix group, namely via the isomorphism $\phi : S^1 \simeq SO(2), \phi(e^{i\theta}) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$.

2.6. If $\Gamma < G$ is a discrete, normal subgroup then $G/\Gamma$ is a Lie group with $\text{Lie}(G/\Gamma) = \mathfrak{g}$. Let $p : G \to G/\Gamma$ the projection onto $\Gamma, p(x) = x\Gamma$. Then $\exp_{G/\Gamma} : \mathfrak{g} \to G/\Gamma$ is given by $\exp_{G/\Gamma} = p \circ \exp_G$.

2.6.1. Example. $\mathbb{Z} < \mathbb{R}$, so $G_6 = \mathbb{R}/\mathbb{Z}$ is a Lie group with Lie algebra $\text{Lie}(G_6) = \mathbb{R}$ and exponential map $\exp(t) = t \pmod{\mathbb{Z}}$. It can be easily verified that the map $\phi : G_6 \simeq S^1, \phi(t) = e^{2\pi it}$ is a Lie group isomorphism.
3. The Torus

3.1. Lattices in vector spaces. Assume $V$ is a finite dimensional vector space over $\mathbb{R}$. A lattice is a discrete (additive) subgroup $L \subset V$ such that the set $L$ spans the vector space $V$ over $\mathbb{R}$.

3.1.1. Lemma. Let $L \subset V$ be a lattice, and $\dim_{\mathbb{R}} V = n$. Then there exist $n$ linearly independent vectors $e_1, \ldots, e_n \in L$ such that

$$L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$$

Proof: for $n = 1$, one chooses $e_1 = \min \{ x : x \in L, x > 0 \}$. Then $L = \mathbb{Z}e_1 \subset \mathbb{R}$. For $n > 1$, we proceed by induction. Put an arbitrary (positive definite) inner product structure on $V$ and choose $e_1 \in L$ such that $\|e_1\| = \min \{ \|x\| : x \in L \}$ (such an $e_1$ exists since $L$ is discrete). Let $W = e_1^\perp \subset V$ and $p : V \to W$ the orthogonal projection. Then $p(W)$ is a lattice in $W$, hence there exist $e_2, \ldots, e_n \in L$ such that $W = \mathbb{Z}p(e_1) \oplus \cdots \oplus \mathbb{Z}p(e_n)$. Then it is not hard to check that $V = \oplus_{1 \leq j \leq n} \mathbb{Z}e_j$.

3.2. Theorem. If $L \subset V$ is a lattice in $V$, then there exists a Lie group isomorphism $V/L \simeq S^1 \times \cdots \times S^1$.

Proof: for $v = \sum x_j e_j \in V$ set $f(v) := (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$.

3.2.1. Notation. $\mathbb{T}^n = S^1 \times \cdots \times S^1$ is the $n$-dimensional (real) torus. We can use $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ as an alternative description.

3.3. Characterization of connected abelian groups. Let $G$ a connected, abelian group. Then $g$ is abelian and the exponential map $\exp : g \to G$ has the property $\exp(X + Y) = \exp(X) \exp(Y)$, $\forall \ X, Y \in g$. Hence $\exp : g \to G$ is a Lie group homomorphism from the vector group $(g, +)$ into $G$. We will need two observations

a) Let $L = \ker(\exp_G) \subset g$. Then $L$ is a discrete (additive) subgroup of $g$. To see this, notice that $\ker(\exp_G) = \{0\}$, where $U$ is neighborhood of $0$ small enough such that $\exp : U \simeq \exp(U)$.

b) The map $\exp : g \to G$ is surjective. This follows from the fact that the image $\exp(g)$ is an open subgroup of $G$ (subgroup for obvious reasons, and open since $\exp$ is a local diffeomorphism) hence also closed. Since $G$ is connected, we must have $\exp(g) = G$.

We conclude that $G \simeq g / \ker(\exp_G) = g / L$. Let now $W := \text{span}_{\mathbb{R}}(L) \leq g$ the vector space spanned by $L$ (over $\mathbb{R}$) and let $d := \dim_{\mathbb{R}} W$. This means that $d = \text{rank}(L)$. Then necessarily $0 \leq d \leq n$. Since $L$ is a lattice in $W$, there exist $e_1, \ldots, e_d \in L$ such that $W = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_d$. Let $f_1, \ldots, f_d$ a basis of the orthogonal complement $W^\perp$. Then $G \simeq g / L = (W \oplus W^\perp) / L = (W / L) \times W^\perp$, which is the product of a torus and a vector group. Explicitly we have an isomorphism $f : G \to \mathbb{T}^d \times \mathbb{R}^{n - d}$ given by

$$f(g) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}, y_{d+1}, \ldots, y_n)$$

We sum up this discussion with the following

3.4. Theorem. Assume $G$ is a connected, abelian Lie group, $\dim_{\mathbb{R}} G = n$. Let $L = \exp^{-1}(e) \subset g$. Then:

a) $G \simeq g / L \simeq \mathbb{T}^d \times \mathbb{R}^{n - d}$, where $d = \text{rank}(L)$.

b) If $G$ is compact, then $G \simeq \mathbb{T}^n$.

3.4.1. Example. For $G = \mathbb{C}^\times, g = C, L = 2\pi i \mathbb{Z}, W = \text{span}_{\mathbb{R}}(L) = i\mathbb{R}, d = 1$, hence $\mathbb{C}^\times \simeq \mathbb{C} / L \simeq S^1 \times \mathbb{R}$. The explicit isomorphism $f : \mathbb{C}^\times \to S^1 \times \mathbb{R}$ is given by $f(z) = (\frac{1}{\pi}, \log(|z|))$.

3.5. Topological generators. By definition, $g \in G$ is a topological generator for $G$ if $G = \langle g \rangle$.

3.5.1. Lemma. Let $K$ a compact, abelian Lie group.

a) If $K$ is connected (torus), then $K$ has a topological generator.

b) If $K / K^0$ is cyclic, then $K$ has a topological generator.

Proof. a) Assume for simplicity that $K = \mathbb{R}^2 / \mathbb{Z}^2$. Kronecker’s theorem: if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is such that $k : \xi \notin \mathbb{Z}, \forall k \in \mathbb{Z}^2$, then $n\xi$ is dense (mod $\mathbb{Z}^2$). This says that
given: $x_1, x_2 \in \mathbb{R}^2, \epsilon > 0$, there exist: $m_1, m_2, n \in \mathbb{Z}$ : $|x_i - m_i - n\xi| < \epsilon, i = 1, 2$

In particular this shows that $g = (\sqrt{2}, \sqrt{3})$ is a topological generator for $\mathbb{R}^2 / \mathbb{Z}^2$.

b): homework.
4. **Compact Lie groups**

4.1. **Definition.** A (closed) Lie subgroup \( G \subseteq GL(n, \mathbb{C}) \) is compact if and only if it is bounded. In other words, if and only if the entries of its elements are uniformly bounded:

\[ \exists C > 0 \text{ (depending on } G \text{) such that: } |g_{ij}| \leq C, \ 1 \leq i,j \leq n, \ \forall g = (g_{ij}) \in G \]

4.1.1. **Note.** Compactness of a Lie group is an intrinsic feature and does not depend on the particular embedding in a \( GL(n, \mathbb{C}) \). This feature is clearly preserved under isomorphisms.

4.2. **Examples.** The following Lie groups are compact: a) \( SO(n) := \{ g \in O(n) : \det g = 1 \} \), \( SU(n) := \{ g \in GL(n, \mathbb{C}) : gg^* = I_n \} \).

Proof in the case c). \( U(n) := \{ g \in GL(n, \mathbb{C}) : g\bar{g}^t = I_n \} \) is a closed (the inverse image of a closed set through a continuous function) subgroup of \( GL(n, \mathbb{C}) \). For \( g \in U(N) \), \( 1 = \sum_{k=1}^N g_{ik} \bar{g}_{ik} = \sum_{k=1}^N |g_{ik}|^2 \), hence \(|g_{ik}| \leq 1\). This shows that \( U(n) \) is compact. Since \( SU(n) = \det^{-1}(1) \subseteq U(n) \) is a closed subset of \( U(n) \), \( SU(n) \) is compact as well.

4.3. **The connected component.** If \( G \) is compact, then \( G^0 \triangleleft G \) is both open and closed in \( G \).

a) In particular, this shows that \( G^0 \) is compact as well.

b) The open covering \( G = \cup_{g \in G} gG^0 \) admits a finite subcover, since \( G \) is compact. That is, there exist finitely many \( g_1, \ldots, g_k \in G \) such that \( G = \cup_{i=1}^k g_i G^0 \). This shows that \( [G : G^0] < +\infty \).

5. **Maximal torus of a compact group**

Throughout this section \( G \) denotes a connected, compact Lie group.

5.1. **Definition.** A maximal torus \( T \subset G \) is a closed subgroup of \( G \) with the following properties:

a) \( T \) is a torus

b) If \( T_1 \subset G \) is another torus such that \( T \subset T_1 \), then \( T = T_1 \).

Maximal tori always exist: let \( X \in \mathfrak{g} \) and set \( H_1 = \exp(tX) \). Then \( H_1 \) is a torus inside \( G \). If it is maximal, then \( T = H_1 \). Otherwise, we obtain an increasing sequence \( H_1 \subsetneq H_2 \subsetneq \ldots \) of tori. But this sequence has to end after finitely many steps, since \( \text{Lie}(H_1) \subsetneq \text{Lie}(H_2) \subsetneq \ldots \) is a sequence of subspaces of \( \mathfrak{g} \) of strictly increasing dimension.

5.2. **Theorem.** Fix a maximal torus \( T \leq G \).

a) For any \( x \in \mathfrak{g} \), there exist \( g \in G \) such that \( x = g^{-1}Tg \).

b) Any two maximal tori are conjugate: if \( S \subset G \) is another maximal torus, there exists \( g \in G \) such that \( S = g^{-1}Tg \).

c) \( T \) is maximal abelian: if \( T \leq S \leq G \) and \( S \) is abelian, then \( T = S \).

5.2.1. **Proof.** a) Idea: the map \( l(x) : G/T \rightarrow G/T, \ l(x) = xgT \) has a fixed point. The technical details (actual counting of fixed points) involves ideas from algebraic topology (Lefschetz trace formula).

b) If \( S \leq G \) is a (any) torus, it has a topological generator \( x \). But then there exists \( g \in G \) such that \( x = g^{-1}Tg \), hence \( S = \langle x \rangle \leq g^{-1}Tg \). If \( S \) is maximal, then \( S = g^{-1}Tg \).

c) We need to prove that \( C_G(T) = T \). Let \( g \in C_G(T) \). Then \( K = \langle g, T \rangle \) is compact, abelian, and \( K/K^0 = \langle g \rangle \) is cyclic. Hence \( K \) has a topological generator. In particular there exists \( h \in G \) such that \( K \leq h^{-1}Th \). Then \( T \leq K \leq h^{-1}Th \). By maximality, \( T = K \). Therefore \( g \in T \).

5.3. **Rank.** Let \( T \subset G \) a maximal torus. By definition, rank(\( G \)) = dim T (does not depend on the choice of \( T \)). Let \( \mathfrak{t} := \text{Lie}(T) \subset \mathfrak{g} \) the Then \( \mathfrak{t} \) is a maximal abelian subalgebra of \( \mathfrak{g} \). One also refers to \( T, \mathfrak{t} \) as the Cartan subgroup (subalgebra) of \( G \), \( \mathfrak{g} \) resp.

5.4. **Examples.**

5.4.1. \( G = SU(2), \ T = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \simeq S^1 \). One reason why \( T \) is a maximal torus is because \( su(2) \) has no 2-dimensional abelian subalgebras. Alternative choice: \( T_2 = SO(2) \). Notice that \( T \) and \( T_2 \) are conjugate in \( SU(2) \) via \( g = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \).

5.4.2. For \( G = SO(4), \ T = SO(2) \times SO(2) \simeq \mathbb{T}^2 \).
6. Lie Group Homomorphisms

6.1. Definition. Assume $G$ and $H$ are Lie groups. A map $f : G \rightarrow H$ satisfying the two properties:

- $f$ is a group homomorphism
- $f$ is a $C^\infty$ map

is called a homomorphism of Lie groups, or simply a homomorphism.

6.2. Second commutative diagram. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of $G$, $H$ resp. For $X \in \mathfrak{g}$, $t \mapsto \phi(\exp(tX))$ is a (continuous) one-parameter subgroup of $H$. Therefore $\exists! Y \in \mathfrak{h}$ such that

$$
\phi(\exp(tX)) = \exp_H(tY), \quad \forall t \in \mathbb{R}
$$

Such an element $Y \in \mathfrak{h}$ is unique and we denote it by $Y = \phi_*(X) = d\phi(X)$. In other words we have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\phi_*} & \mathfrak{h} \\
\exp_G \downarrow & & \downarrow \exp_H \\
G & \xrightarrow{\phi} & H
\end{array}
$$

6.3. Theorem. The map $\phi_* : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

6.3.1. Proof. By definition

$$
\phi_*(X) = \frac{d}{dt}\phi(\exp tX)|_{t=0}
$$

We will use this formula to determine the properties of $\phi_*$.

Additivity. Recall:

$$
M(tX,tY) = \exp^{-1}(\exp(tX)\exp(tY)) = tX + tY + O(t^2) \Rightarrow \exp(tX)\exp(tY) = \exp(tX + tY + O(t^2))
$$

Therefore:

$$
\frac{d}{dt} \phi(\exp(tX + Y)|_{t=0} = \frac{d}{dt} \phi(\exp(tX + O(t^2)))|_{t=0}
$$

$$
= \frac{d}{dt} \phi(\exp(tX) \cdot \exp(tY)|_{t=0} = \frac{d}{dt} \phi(\exp(tX) \cdot \phi(tY)|_{t=0}
$$

$$
= \frac{d}{dt} \phi(\exp(tX)|_{t=0} + \frac{d}{dt} \phi(tY)|_{t=0} \quad \text{[Leibniz rule]}
$$

$$
= \phi_*(X) + \phi_*(Y)
$$

Bracket relations. First, we prove that $\phi_*(\text{Ad}(g)X) = \text{Ad}(\phi(g))(\phi_*(X))$. To see this

$$
\frac{d}{dt} \phi(\exp t \text{Ad}(g)X)|_{t=0} = \frac{d}{dt} \phi(g)(\phi(tX)\phi(g)|_{t=0}
$$

$$
= \frac{d}{dt} \phi(g) \exp(t\phi_*(X)) \phi(g^{-1})|_{t=0} = \frac{d}{dt} \exp(t \text{Ad}(\phi(g))\phi_*(X))|_{t=0} = \text{Ad}(\phi(g)) \phi_*(X)
$$

For $g = \exp tY$ this is $\phi_*(\text{Ad}(\exp tY)X) = \text{Ad}(\phi(\exp tY))(\phi_*(X))$. Since $\text{Ad}(\exp tY) = e^{t\text{ad}(Y)}$ this means

$$
\phi_*(e^{t\text{ad}(Y)}X) = \text{Ad}(\exp t\phi_*(Y))\phi_*(X) = e^{t\text{ad} \phi_*(Y)} \phi_*(X)
$$

Hence

$$
\phi_*(X + t \text{ad}(Y)X + \frac{1}{2} t^2 \text{ad}(Y)^2 X + \ldots) = \phi_*(X) + t \text{ad}(\phi_*(Y))\phi_*(X) + \frac{1}{2} t^2 \text{ad}(\phi_*(Y))^2 \phi_*(X) + \ldots
$$

Differentiating at $t = 0$ on both sides we obtain $\phi_*(\text{ad}(Y)X) = \text{ad}(\phi_*(Y)) \phi_*(X)$.

7. Finite Dimensional Representations of Lie Groups

7.1. Definition. Assume $V$ is a finite dimensional vector space over $\mathbb{C}$ and $G$ a Lie group. A representation $(\pi, V)$ of $G$ on $V$ is a Lie group homomorphism

$$
\pi : G \rightarrow GL(V)
$$
7.2. **Action of Lie algebra.** Associated to the representation \((\pi, V)\) there is a Lie algebra homomorphism \(\pi_* : \mathfrak{g} \to \mathfrak{gl}(V)\), such that the diagram is commutative

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\pi_*} & \mathfrak{gl}(V) \\
\exp & \downarrow & \downarrow \exp \\
G & \xrightarrow{\phi} & GL(V)
\end{array}
\]

We say that the datum of a Lie algebra homomorphism \(\pi_* : \mathfrak{g} \to \mathfrak{gl}(V)\) determines a *Lie algebra representation* of \(\mathfrak{g}\) on the vector space \(V: \mathfrak{g} \times V \to V, (X,v) \mapsto \pi_*(X)v\). Equivalently, \(V\) is a \(\mathfrak{g}\)-module.

7.3. **Example: the adjoint representation.** Associated to an arbitrary Lie group \(G\) one has the adjoint representation \(\text{Ad} : G \to \mathfrak{gl}(\mathfrak{g})\). Then the associated Lie algebra action is \((\text{Ad})_* = \text{ad}\).