8. COMPACT LIE GROUPS AND REPRESENTATIONS

1. Abelian Lie groups

1.1. **Theorem.** Assume G is a Lie group and $\mathfrak{g}$ its Lie algebra. Then

$$G^0 \text{ is abelian iff } \mathfrak{g} \text{ is abelian}$$

1.1.1. *Proof.* "$\Leftarrow$". Let $0 \in U \subset \mathfrak{g}$ and $e \in V \subset G$ small (symmetric) neighborhoods of 0 in $\mathfrak{g}$ and resp. of e in G such that $\exp : U \simeq V$. Pick $x, y \in V$ and let $x = \exp X$, $y = \exp Y$, with $X, Y \in U$. Then

$$xy = \exp X \exp Y = \exp M(X,Y) = \exp(X+Y) = \exp(Y)\exp(X) = yx$$

Since V generates G^0, it follows that G^0 is abelian.

"$\Rightarrow$". Let $g \in G^0$ and $Y \in \mathfrak{g}$. Since $\exp(tY) \in G^0$, $g\exp(tY)g^{-1} = \exp(tY)$. Differentiating this identity at $t = 0$ we obtain $\mathrm{Ad}(g)Y = Y$.

Let now $X, Y \in \mathfrak{g}$. Since $\exp(tX) \in G^0$, $\mathrm{Ad}(\exp(tX))Y = Y$. Differentiating the identity at $t = 0$ we obtain $\mathrm{ad}(X)Y = 0$, so $\mathfrak{g}$ is abelian.

1.2. **Example.** The Lie algebra of $O(2)$ is abelian (one-dimensional), yet $O(2)$ is not abelian:

$$\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \neq \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$$

However its connected component $SO(2)$ is connected.

2. Explicit examples of abelian Lie groups

2.1. $G_1 = \mathbb{R}_{>0}$. The group operation is $x \cdot y = xy$. Then $G_1 = G^0$, where $G = GL(1,\mathbb{R})$. Hence $Lie(G_1) = Lie(G^0) = gl(1,\mathbb{R}) \simeq \mathbb{R}$ and $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is given by $\exp(t) = e^t$.

$$\begin{tikzcd}
gl(1,\mathbb{R}) \arrow[d, "\exp"'] & \\
G_1 \arrow[r, phantom, "\subset"] & GL(1,\mathbb{R})
\end{tikzcd}$$

2.2. $G_2 = (\mathbb{R}, +)$. The group structure is additive $x \cdot y = x + y$. We realize G_2 as a matrix Lie group via the group isomorphism $\phi : G_2 \to G_1$, $\phi(x) = e^x$. Hence $Lie(G_2) = Lie(G_1) = \mathbb{R}$ and the exponential map pulls back:

$$\exp_{G_2} : \mathbb{R} \to G_2, \quad \phi \circ \exp_{G_2} = \exp_{G_1}$$

So it simply given by $\exp_{G_2} : \mathbb{R} \to \mathbb{R}$, $\exp_{G_2}(t) = t$.

2.3. In general, $Lie(G \times H) = \mathfrak{g} \times \mathfrak{h}$, and $\exp_{G\times H}(X,Y) = (\exp_G(X), \exp_H(Y))$. In particular, for any finite dimensional real vector space V, $G_3 = (V,+)$ is a Lie group with abelian Lie algebra $Lie((V,+)) = (V, [\cdot,\cdot]_{ab})$ and exponential map $\exp_V = id_V : V \to V$. Such an abelian Lie group is called a vector group.

2.4. Similarly $G_4 := \mathbb{C}^\times \subset GL(1,\mathbb{C})$ is a Lie group with Lie algebra $Lie(G_4) = \mathbb{C}$ and exponential map $\exp_{G_4} : \mathbb{C} \to \mathbb{C}^\times$, $\exp_{G_4}(z) = e^z$.

2.5. $G_5 = \mathbb{S}^1 := \{e^{i\theta} : \theta \in \mathbb{R}\} \subset \mathbb{C}^\times$. Then $Lie(\mathbb{S}^1) = i\mathbb{R} \subset \mathbb{C} = Lie(\mathbb{C}^\times)$ and $\exp_{\mathbb{S}^1} : i\mathbb{R} \to \mathbb{S}^1$ is the usual exponential. Equivalently, we can think of $Lie(\mathbb{S}^1) = \mathbb{R}$ with $\exp_{\mathbb{S}^1}(t) = e^{it}$.

There is an alternative realization of $\mathbb{S}^1$ as a matrix group, namely via the isomorphism $\phi : \mathbb{S}^1 \simeq SO(2)$, $\phi(e^{i\theta}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

2.6. If $\Gamma \lhd G$ is a discrete, normal subgroup then G/Γ is a Lie group with $Lie(G/\Gamma) = \mathfrak{g}$. Let $p : G \to G/\Gamma$ the projection onto Γ, $p(x) = x\Gamma$. Then $\exp_{G/\Gamma} : \mathfrak{g} \to G/\Gamma$ is given by $\exp_{G/\Gamma} = p \circ \exp_G$.

2.6.1. *Example.* $\mathbb{Z} \lhd \mathbb{R}$, so $G_6 = \mathbb{R}/\mathbb{Z}$ is a Lie group with Lie algebra $Lie(G_6) = \mathbb{R}$ and exponential map $\exp(t) = t (\mathrm{mod}\, \mathbb{Z})$. It can be easily verified that the map $\phi : G_6 \simeq \mathbb{S}^1$, $\phi(t) = e^{2\pi i t}$ is a Lie group isomorphism.

3. The Torus

3.1. **Lattices in vector spaces.** Assume V is a finite dimensional vector space over $\mathbb{R}$. A *lattice* is a discrete (additive) subgroup $L \subset V$ such that the set L spans the vector space V over $\mathbb{R}$.

3.1.1. *Lemma.* Let $L \subset V$ is a lattice, and $\dim_{\mathbb{R}} V = n$. Then there exist n linearly independent vectors $e_1, \dots, e_n \in L$ such that

$$L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$$

Proof: for $n = 1$, one chooses $e_1 = \min\{x : x \in L, x > 0\}$. Then $L = \mathbb{Z}e_1 \subset \mathbb{R}$. For $n > 1$, we proceed by induction. Put an arbitrary (positive definite) inner product structure on V and choose $e_1 \in L$ such that $\|e_1\| = \min\{\|x\| : x \in L\}$ (such an e_1 exists since L is discrete). Let $W = e_1^{\perp} \subset V$ and $p : V \to W$ the orthogonal projection. Then $p(W)$ is a lattice in W, hence there exist $e_2, \dots, e_n \in L$ such that $W = \mathbb{Z}p(e_1) \oplus \cdots \oplus \mathbb{Z}p(e_n)$. Then it is not hard to check that $V = \oplus_{1 \le j \le n} \mathbb{Z}e_j$.

3.2. **Theorem.** If $L \subset V$ is a lattice in V, then there exists a Lie group isomorphism $V/L \simeq \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Proof: for $v = \sum x_j e_j \in V$ set $f(v) := (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$.

3.2.1. *Notation.* $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is the n-dimensional (real) torus. We can use $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ as an alternative description.

3.3. **Characterization of connected abelian groups.** Let G a connected, abelian group. Then $\mathfrak{g}$ is abelian and the exponential map $\exp : \mathfrak{g} \to G$ has the property $\exp(X+Y) = \exp(X)\exp(Y)$, $\forall\, X, Y \in \mathfrak{g}$. Hence $\exp : \mathfrak{g} \to G$ is a Lie group homomorphism from the vector group $(\mathfrak{g}, +)$ into G. We will need two observations

a) Let $L = \ker(\exp_G) \subset \mathfrak{g}$. Then L is a discrete (additive) subgroup of $\mathfrak{g}$. To see this, notice that $L \cap U = \{0\}$, where U is neighborhood of 0 small enough such that $\exp : U \simeq \exp(U)$.

b) The map $\exp : \mathfrak{g} \to G$ is surjective. This follows from the fact that the image $\exp(\mathfrak{g})$ is an *open subgroup* of G (subgroup for obvious reasons, and open since exp is a local diffeomorphism) hence also *closed.* Since G is connected, we must have $\exp(\mathfrak{g}) = G$.

We conclude that $G \simeq \mathfrak{g}/\ker(\exp) = \mathfrak{g}/L$. Let now $W := \mathit{span}_{\mathbb{R}}(L) \le \mathfrak{g}$ the vector space spanned by L (over $\mathbb{R}$) and let $d := \dim_{\mathbb{R}} W$. This means that $d = \operatorname{rank}(L)$. Then necessarily $0 \le d \le n$. Since L is a lattice in W, there exist $e_1, \dots, e_d \in L$ such that $W = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_d$. Let $f_{d+1}, \dots, f_n$ a basis of the orthogonal complement $W^{\perp}$. Then $G \simeq \mathfrak{g}/L = (W \oplus W^{\perp})/L = (W/L) \times W^{\perp}$, which is the product of a torus and a vector group. Explicitly we have an isomorphism $f : G \to \mathbb{T}^d \times \mathbb{R}^{n-d}$ given by

$$\text{for} \quad g = \exp_G\{\sum_{j=1}^{d} x_j e_j + \sum_{k=d+1}^{n} y_k f_k\}, \quad f(g) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_d}, y_{d+1}, \dots, y_n)$$

We sum up this discussion with the following

3.4. **Theorem.** Assume G is a connected, abelian Lie group, $\dim G = n$. Let $L = \exp^{-1}(e) \subset \mathfrak{g}$. Then:

a) $G \simeq \mathfrak{g}/L \simeq \mathbb{T}^d \times \mathbb{R}^{n-d}$, where $d = \operatorname{rank}(L)$.

b) If G is compact, then $G \simeq \mathbb{T}^n$.

3.4.1. *Example.* For $G = \mathbb{C}^{\times}$, $\mathfrak{g} = \mathbb{C}$, $L = 2\pi i\mathbb{Z}$, $W = \mathit{span}_{\mathbb{R}}(L) = i\mathbb{R}$, $d = 1$, hence $\mathbb{C}^{\times} \simeq \mathbb{C}/L \simeq \mathbb{S}^1 \times \mathbb{R}$. The explicit isomorphism $f : \mathbb{C}^{\times} \to \mathbb{S}^1 \times \mathbb{R}$ is given by $f(z) = (\frac{z}{|z|}, \log|z|)$.

3.5. **Topological generators.** By definition, $g \in G$ is a topological generator for G if $G = \overline{< g >}$.

3.5.1. *Lemma.* Let K a compact, abelian Lie group.

a) If K is connected (torus), then K has a topological generator.

b) If K/K^0 is cyclic, then K has a topological generator.

Proof. a) Assume for simplicity that $K = \mathbb{R}^2/Z^2$. Kronecker's theorem: if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is such that $k \cdot \xi \notin \mathbb{Z}$, $\forall 0 \ne k \in \mathbb{Z}^d$, then $n\xi$ is dense $(\operatorname{mod} \mathbb{Z}^2)$. This says that

$$\text{given:} \quad x_1, x_2 \in \mathbb{R}^2, \epsilon > 0, \quad \text{there exist:} \quad m_1, m_2, n \in \mathbb{Z} : \quad |x_i - m_i - n\xi_i| < \epsilon, i = 1, 2$$

In particular this shows that $g = (\sqrt{2}, \sqrt{3})$ is a topological generator for $\mathbb{R}^2/\mathbb{Z}^2$.

b): homework.

4. Compact Lie groups

4.1. **Definition.** A (closed) Lie subgroup $G \subseteq GL(n,\mathbb{C})$ is *compact* if and only if it is bounded. In other words, if and only if the entries of its elements are uniformly bounded:

$$\exists C > 0 \text{ (depending on } G\text{) such that: } \quad |g_{ij}| \leq C,\ 1 \leq i,j \leq n, \quad \forall g = (g_{ij}) \in G$$

4.1.1. *Note.* Compactness of a Lie group is an intrinsic feature and does not depend on the particular embedding in a $GL(n,\mathbb{C})$. This feature is clearly preserved under isomorphisms.

4.2. **Examples.** The following Lie groups are compact: a) $\mathbb{T}^d$; b) $O(n) := \{g \in GL(n,\mathbb{R}) : gg^t = I_n\}$, $SO(n) := \{g \in O(n) : \det g = 1\}$; c) $U(n) := \{g \in GL(n,\mathbb{C}) : gg^* = I_n\}$, $SU(n) := \{g \in U(n) : \det g = 1\}$.

Proof in the case c). $U(n) := \{g \in GL(n,\mathbb{C}) : g\bar{g}^t = I_n\}$ is a closed (the inverse image of a closed set through a continuous function) subgroup of $GL(n,\mathbb{C})$. For $g \in U(N)$, $1 = \sum_{k=1}^N g_{ik}\bar{g}_{ik} = \sum_{k=1}^N |g_{ik}|^2$, hence $|g_{ik}| \leq 1$. This shows that $U(n)$ is compact. Since $SU(n) = \det^{-1}(1) \subset U(n)$ is a closed subset of $U(n)$, $SU(n)$ is compact as well.

4.3. **The connected component.** If G is compact, then $G^0 \lhd G$ is both open and closed in G.

a) In particular, this shows that G^0 is compact as well.

b) The *open* covering $G = \cup_{g\in G} gG^0$ admits a finite subcover, since G is compact. That is, there exist finitely many $g_1, \ldots, g_k \in G$ such that $G = \sqcup_{i=1}^k g_i G^0$. This shows that $[G : G^0] < +\infty$.

5. Maximal torus of a compact group

Throughout this section G denotes a *connected, compact* Lie group.

5.1. **Definition.** A maximal torus $T \subset G$ is a closed subgroup of G with the following properties:

a) T is a torus

b) If $T_1 \subset G$ is another torus such that $T \subset T_1$, then $T = T_1$.

Maximal tori always exist: let $X \in \mathfrak{g}$ and set $H_1 = \overline{\exp(tX)}$. Then H_1 is a torus inside G. If it is maximal, then set $T = H_1$. Otherwise, we obtain an increasing sequence $H_1 \subsetneqq H_2 \subsetneqq \ldots$ of tori. But this sequence has to end after finitely many steps, since $Lie(H_1) \subsetneqq Lie(H_2) \subsetneqq \ldots$ is a sequence of subspaces of $\mathfrak{g}$ of strictly increasing dimension.

5.2. **Theorem.** Fix a maximal torus $T \leq G$.

a) For any $x \in G$, there exist $g \in G$ such that $x \in g^{-1}Tg$.

b) Any two maximal tori are conjugate: if $S \subset G$ is another maximal torus, there exists $g \in G$ such that $S = g^{-1}Tg$.

c) T is maximal abelian: if $T \leq S \leq G$ and S is abelian, then $T = S$.

5.2.1. *Proof.* a) Idea: the map $l(x) : G/T \to G/T$, $l(x)(gT) = xgT$ has a fixed point. The technical details (actual counting of fixed points) involves ideas from algebraic topology (Lefschetz trace formula).

b) If $S \leq G$ is a (any) torus, it has a topological generator x. But then there exists $g \in G$ such that $x \in g^{-1}Tg$, hence $S = \overline{<x>} \subset g^{-1}Tg$. If S is maximal, then $S = g^{-1}Tg$.

c) We need to prove that $C_G(T) = T$. Let $g \in C_G(T)$. Then $K = \overline{<g,T>}$ is compact, abelian, and $K/K^0 = <g>$ is cyclic. Hence K has a topological generator. In particular there exists $h \in G$ such that $K \subset h^{-1}Th$. Then $T \subseteq K \subseteq h^{-1}Th$. By maximality, $T = K$. Therefore $g \in T$.

5.3. **Rank.** Let $T \subset G$ a maximal torus. By definition, $\text{rank}(G) = \dim T$ (does not depend on the choice of T). Let $\mathfrak{t} := Lie(T) \subset \mathfrak{g}$ the Then $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$. One also refers to $T, \mathfrak{t}$ as the Cartan subgroup (subalgebra) of G, $\mathfrak{g}$ resp.

5.4. **Examples.**

5.4.1. $G = SU(2)$, $T = \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix} \simeq S^1$. One reason why T is a maximal torus is because $su(2)$ has no 2-dimensional abelian subalgebras. Alternative choice: $T_2 = SO(2)$. Notice that T and T_2 are conjugate in $SU(2)$ via $g = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$.

5.4.2. For $G = SO(4)$, $T = SO(2) \times SO(2) \simeq \mathbb{T}^2$.

6. Lie group homomorphisms

6.1. **Definition.** Assume G and H are Lie groups. A map $f : G \to H$ satisfying the two properties:

- f is a group homomorphism
- f is a C^∞ map

is called a homomorphism of Lie groups, or simply a homomorphism.

6.2. **Second commutative diagram.** Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H resp. For $X \in \mathfrak{g}$, $t \mapsto \phi(\exp(tX))$ is a (continuous) one-parameter subgroup of H. Therefore $\exists! Y \in \mathfrak{h}$ such that

$$\phi(\exp_G(tX)) = \exp_H(tY), \quad \forall t \in \mathbb{R}$$

Such an element $Y \in \mathfrak{h}$ is unique and we denote it by $Y = \phi_*(X) = d\phi(X)$. In other words we have a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi_*"] \arrow[d, "\exp_G"'] & \mathfrak{h} \arrow[d, "\exp_H"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

6.3. **Theorem.** The map $\phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

6.3.1. *Proof.* By definition

$$\boxed{\phi_*(X) = \tfrac{d}{dt}\phi(\exp tX)|_{t=0}}$$

We will use this formula to determine the properties of ϕ_*.

Additivity. Recall:

$$M(tX, tY) = \exp^{-1}(\exp(tX)\exp(tY)) = tX + tY + O(t^2) \Rightarrow \exp(tX)\exp(tY) = \exp(tX + tY + O(t^2))$$

Therefore:

$$\begin{aligned} &\frac{d}{dt}\,\phi(\exp t(X+Y))|_{t=0} = \frac{d}{dt}\,\phi(\exp\{t(X+Y) + O(t^2)\})|_{t=0} \\ &= \frac{d}{dt}\,\phi(\exp tX \cdot \exp tY)|_{t=0} = \frac{d}{dt}\,\phi(\exp tX)\cdot\phi(\exp tY)|_{t=0} \\ &= \frac{d}{dt}\,\phi(\exp tX)|_{t=0} + \frac{d}{dt}\,\phi(\exp tY)|_{t=0} \qquad \text{[Leibniz rule]} \\ &= \phi_*(X) + \phi_*(Y) \end{aligned}$$

Bracket relations. First, we prove that $\phi_*(\mathrm{Ad}(g)X) = \mathrm{Ad}(\phi(g))(\phi_*(X))$. To see this

$$\begin{aligned} &\frac{d}{dt}\,\phi(\exp t\,\mathrm{Ad}(g)X)|_{t=0} = \frac{d}{dt}\,\phi(g)\,\phi(\exp tX)\phi(g)|_{t=0} \\ &= \frac{d}{dt}\,\phi(g)\exp(t\phi_*(X))\,\phi(g^{-1})|_{t=0} = \frac{d}{dt}\,\exp\Big(t\,\mathrm{Ad}(\phi(g))\phi_*(X)\Big)\Big|_{t=0} = \mathrm{Ad}(\phi(g))\,\phi_*(X) \end{aligned}$$

For $g = \exp tY$ this is $\phi_*(\mathrm{Ad}(\exp tY)X) = \mathrm{Ad}(\phi(\exp tY))\phi_*(X)$. Since $\mathrm{Ad}(\exp tY) = e^{t\,\mathrm{ad}(Y)}$ this means

$$\phi_*(e^{t\,\mathrm{ad}(Y)}X) = \mathrm{Ad}(\exp t\phi_*(Y))\phi_*(X) = e^{t\,\mathrm{ad}\,\phi_*(Y)}\phi_*(X)$$

Hence

$$\phi_*(X + t\,\mathrm{ad}(Y)X + \frac{1}{2}t^2\,\mathrm{ad}(Y)^2X + \dots) = \phi_*(X) + t\,\mathrm{ad}(\phi_*(Y))\phi_*(X) + \frac{1}{2}t^2\,\mathrm{ad}(\phi_*(Y))^2\phi_*(X) + \dots$$

Differentiating at $t = 0$ on both sides we obtain $\phi_*(\mathrm{ad}(Y)X) = \mathrm{ad}(\phi_*(Y))\,\phi_*(X)$.

7. Finite dimensional representations of Lie groups

7.1. **Definition.** Assume V is a finite dimensional vector space over $\mathbb{C}$ and G a Lie group. A representation (π, V) of G on V is a Lie group homomorphism

$$\pi : G \to GL(V)$$

7.2. **Action of Lie algebra.** Associated to the representation (π, V) there is a Lie algebra homomorphism $\pi_* : \mathfrak{g} \to gl(V)$, such that the diagram is commutative

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\pi_*"] \arrow[d, "\exp"'] & gl(V) \arrow[d, "\exp"] \\
G \arrow[r, "\phi"] & GL(V)
\end{tikzcd}$$

We say that the datum of a Lie algebra homomorphism $\pi_* : \mathfrak{g} \to gl(V)$ determines a *Lie algebra representation* of $\mathfrak{g}$ on the vector space V: $\mathfrak{g} \times V \to V$, $(X, v) \mapsto \pi_*(X)v$. Equivalently, V is a $\mathfrak{g}$-module.

7.3. **Example: the adjoint representation.** Associated to an arbitrary Lie group G one has the adjoint representation $\mathrm{Ad} : G \to gl(\mathfrak{g})$. Then the associated Lie algebra action is $(\mathrm{Ad})_* = \mathrm{ad}$.