9. THE HAAR MEASURE AND PETER-WEYL THEOREM

1. The Haar Measure

1.1. Definition. For a topological group, there exists a measure \( \mu_{\text{Haar}} \) or simply \( \mu \), which is left \( G \)-invariant. Formulated simplistically, this is a countably additive function

\[
\mu : \{ \text{open sets of } G \} \to (0, \infty)
\]

with the invariance property \( \mu(gA) = \mu(A) \), for all \( A \subset G \) and \( g \in G \). Here \( gA = \{ gx : g \in A \} \subset G \).

Moreover, \( \mu \) is unique up to a scalar multiple. We fix from now on the Haar measure \( \mu \). For \( A \subset G \) an open set, we may refer to \( \mu(A) \) also as \( \text{vol}(A) \).

1.2. Theorem. \( G \) is compact if and only if \( \text{vol}(G) < +\infty \).

1.3. Integral. For \( \phi : G \to \mathbb{C} \) a continuous function (vanishing outside a compact set), the integral \( \int_G \phi d\mu(g) \) or simply \( \int_G \phi dg \) can be defined as the limit of the Riemann sums

\[
\sum_{i=1}^{N} \phi(\xi_i)\mu(A_i)
\]

where \( A_i \) is a partition of \( G \) and \( \xi_i \in A_i \). In practice, we will use

- \( \int_G (c_1 \phi_1 + \phi_2) dg = c_1 \int_G \phi_1 dg + \int_G \phi_2 dg \), \( c_1, c_2 \in \mathbb{C}, \phi_1, \phi_2 : G \to \mathbb{C} \)
- \( \int_G \phi(hg) dg = \int_G \phi dg \), \( \forall h \in G, \phi : G \to \mathbb{C} \)

1.3.1. Note. We will determine an explicit formula of the Haar measure (Weyl’s integration formula) for some special compact groups. However the very existence of the Haar measure is important for many purposes.

1.4. Modulus function. For a fixed \( g \in G \), it is easy to check that \( \mu^g(A) := \mu(\{Ag\}) \) defines a left-invariant measure on \( G \). By the uniqueness of Haar measure, there must exists \( \delta(g) > 0 \) such that \( \mu^g(A) = \delta(g) \mu(A) \), for any open set \( A \). It is easy to check that \( \delta(gh) = \delta(g)\delta(h) \), hence we have a (continuous) homomorphism

\[
\delta : G \to \mathbb{R}_{>0}
\]

In particular \( \int_G \phi(xg) dx = \delta(g)^{-1} \int_G \phi(x) dx \), \( \forall g \in G \).

1.4.1. Definition. A group \( G \) is unimodular if \( \delta(g) = 1 \), \( \forall g \in G \). In other words, a group is unimodular if the Haar measure (by definition left-invariant) is right-invariant as well.

1.5. Theorem. A group of the following type is necessarily unimodular:

a) abelian groups
b) compact groups
c) \( SL(2, \mathbb{R}) \)

1.5.1. Proof. b) If \( G \) is a compact group, the image \( \delta(G) \) is a compact subgroup of \( (0, \infty) \), in particular bounded. Hence \( \delta(G) = \{ 1 \} \). c) For \( SL(2, \mathbb{R}) \), consider the differential \( \delta_* : sl(2, \mathbb{R}) \to \mathbb{R} \). Since \( \mathbb{R} \) is abelian, \( \delta_*([X,Y]) = 0 \), \( \forall X, Y \in sl(2, \mathbb{R}) \). But every element of \( sl(2, \mathbb{R}) \) is of this form (check!), hence \( \delta_* \equiv 0 \). Since \( SL(2, \mathbb{R}) \) is connected this determines \( \delta \equiv 1 \).

2. Examples

2.1. For \( G = \mathbb{R}_{>0} \), let \( d_{\text{Haar}}(x) = \sigma(x) dx \), where \( \sigma(x) \) is some function on \( G \). Normalize the Haar measure by \( \sigma(1) = 1 \). Then \( \sigma(ax)d(ax) = \sigma(x) dx \Rightarrow \sigma(a) = \sigma(1) = 1 \), hence \( d_{\text{Haar}}x = \frac{dx}{x} \) on \( \mathbb{R}_{>0} \). Clearly this is a unimodular group (abelian).
Theorem. 2.2.1. by so that \( \text{vol}(G) = \int_G d\mu \). To compute the modulus, let 
\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \\ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \\ \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ \end{bmatrix},
\]
then \( |a|\text{vol}(G) = \int_G d\mu \). This shows that \( G \) is not unimodular. In particular, for \( g = \begin{bmatrix} a & b \\ 0 & 1 \\ \end{bmatrix} \) we have

\[
\int_G \frac{d\mu(h)}{g(h)d\mu} = \int_G \phi(h) \phi(g) d\mu = \int_G \frac{d\mu(h)}{g(h)d\mu} = \int_G \phi(h) d\mu = |a| \int_G \phi(h) d\mu
\]

2.2.1. Let \( B_n \) the upper triangular subgroup of \( GL(n, \mathbb{R}) \). Then

\[
dg = \prod_{1 \leq i \leq j \leq n} \frac{dx_{ij}}{\prod_{k=1}^{n} x_{kk}^2}, \quad \delta_{B_n}(g) = x_{11}^{n-1} x_{22}^{3-n} \ldots x_{nn}^{n-1} = \det(g)^{n-1} \prod_{k=1}^{n} x_{kk}^{2k}
\]

3. Compact groups: Peter-Weyl Theorem

3.1. Theorem.

a) [Finite dimensionality] An irreducible representation of \( G \) is necessarily finite-dimensional.

b) [Weyl trick] Every (finite dimensional) representation \((\pi, V)\) of \( G \) is unitary with respect to some inner product structure on \( V \). In particular, every representation is completely reducible.

c) [Countable dual] The set \( \hat{G} \) is countable.

3.2. Peter Weyl Theorem. We have an equivalence of \( G \times G \) representations:

\[
L^2(G) = \bigoplus_{a \in \hat{G}} V_a^* \otimes V_a
\]

Since the matrix coefficients satisfy the same orthogonality relations as in the finite groups case, an equivalent statement is the following

3.3. Theorem.

a) The set \( \left\{ \delta_{a}^{1/2} \phi_{aij} : a \in \hat{G}, 1 \leq i, j \leq d_a \right\} \) is an orthonormal (Hilbert) basis for \( L^2(G) \).

b) The set \( \left\{ \chi_a : a \in \hat{G} \right\} \) is an orthonormal (Hilbert) basis for \( L^2_{\text{class}}(G) \).

c) Let \( \pi \) a finite dimensional representation of \( G \). Then \( m_{\pi}(\pi) = \langle \chi_{\pi}, \chi_{\pi} \rangle, \forall a \in \hat{G} \).

In particular, \( \pi \) is irreducible if and only if \( \| \chi_{\pi} \|_{L^2(G)} = 1 \).

b) Fourier inversion formula: \( f(g) = \sum_{a \in \hat{G}} d_a \text{trace} \{ \pi_a(x)^* \pi_a(f) \}, \forall f \in C^\infty(G), \forall g \in G \).

b) Parseval identity: \( \int_G |f(g)|^2 dg = \sum_{a \in \hat{G}} d_a^2 \| \pi_a(f) \|^2_{HS} = \sum_{a} d_a \sum_{i,j} |\pi_a(f)_{i,j}|^2, \forall f \in C(G) \).
3.4. Example: $G = S^1$. Here $\hat{G} = G^* = \{\chi_n : n \in \mathbb{Z}\}$, where $\chi_n(e^{i\theta}) = e^{in\theta}$. Then

$$\chi_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = \hat{f}(-n)$$

and hence the Fourier inversion and the Parseval formulas are

$$f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{in\theta}, \quad \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$