9. THE HAAR MEASURE AND PETER-WEYL THEOREM

1. The Haar measure

1.1. **Definition.** For G a topological group, there exists a measure μ_{Haar} or simply μ, which is left G-invariant. Formulated simplistically, this is a countably additive function

$$\mu : \{\text{open sets of } G\} \to (0, \infty)$$

with the invariance property $\mu(gA) = \mu(A)$, for all $A \subset G$ and $g \in G$. Here $gA = \{gx : g \in A\} \subset G$.

Moreover, μ is unique up to a scalar multiple. We fix from now on the Haar measure μ. For $A \subset G$ an open set, we may refer to $\mu(A)$ also as $\mathrm{vol}(A)$.

1.2. **Theorem.** G is compact if and only if $\mathrm{vol}(G) < +\infty$.

1.3. **Integral.** For $\phi : G \to \mathbb{C}$ a continuous function (vanishing outside a compact set), the integral $\int_G \phi d\mu(g)$ or simply $\int_G \phi dg$ can be defined as the limit of the Riemann sums

$$\sum_{i=1}^{N} \phi(\xi_i)\mu(A_i)$$

where A_i is a partition of G and $\xi_i \in A_i$. In practice, we will use

- $\int_G (c_1\phi_1 + \phi_2)dg = c_1 \int_G \phi_1 dg + \int_G \phi_2 dg, \quad c_1, c_2 \in \mathbb{C}, \phi_1, \phi_2 : G \to \mathbb{C}$
- $\int_G \phi(hg)dg = \int_G \phi dg, \quad \forall h \in G, \phi : G \to \mathbb{C}$

1.3.1. *Note.* We will determine an explicit formula of the Haar measure (Weyl's integration formula) for some special compact groups. However the very existence of the Haar measure is important for many purposes.

1.4. **Modulus function.** For a fixed $g \in G$, it is easy to check that $\mu^g(A) := \mu(Ag)$ defines a left-invariant measure on G. By the uniqueness of Haar measure, there must exists $\delta(g) > 0$ such that $\mu^g(A) = \delta(g)\mu(A)$, for any open set A. It is easy to check that $\delta(gh) = \delta(g)\delta(h)$, hence we have a (continuous) homomorphism

$$\delta : G \to \mathbb{R}_{>0}$$

In particular $\int_G \phi(xg)dx = \delta(g)^{-1} \int_G \phi(x)dx$, $\forall g \in G$.

1.4.1. *Definition.* A group G is *unimodular* if $\delta(g) = 1$, $\forall g \in G$. In other words, a group is unimodular if the Haar measure (by definition left-invariant) is right-invariant as well.

1.5. **Theorem.** A group of the following type is necessarily unimodular:
a) abelian groups
b) compact groups
c) $SL(2, \mathbb{R})$

1.5.1. *Proof.* b) If G is a compact group, the image $\delta(G)$ is a compact subgroup of $(0, \infty)$, in particular bounded. Hence $\delta(G) = \{1\}$. c) For $SL(2, \mathbb{R})$, consider the differential $\delta_* : sl(2, \mathbb{R}) \to \mathbb{R}$. Since $\mathbb{R}$ is abelian, $\delta_*([X, Y]) = 0$, $\forall X, Y \in sl(2, \mathbb{R})$. But every element of $sl(2, \mathbb{R})$ is of this form (check!), hence $\delta_* \equiv 0$. Since $SL(2, \mathbb{R})$ is connected this determines $\delta \equiv 1$.

2. Examples

2.1. For $G = \mathbb{R}_{>0}$, let $d_{Haar}(x) = \sigma(x)dx$, where $\sigma(x)$ is some function on G. Normalize the Haar measure by $\sigma(1) = 1$. Then $\sigma(ax)d(ax) = \sigma(x)dx \Rightarrow \sigma(a)a = \sigma(1) = 1$, hence $d_{Haar}x = \frac{dx}{x}$ on $\mathbb{R}_{>0}$. Clearly this is a unimodular group (abelian).

2.2. For $G = \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \subset GL(2,\mathbb{R})$, let $d_{Haar}(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}) = \sigma(x,y)dxdy$. Since $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & ay+b \\ 0 & 1 \end{bmatrix}$, left-invariance of Haar measure means $\sigma(ax, ay+b)a^2dxdy = \sigma(x,y)dxdy$, hence for $x = 1$ and $y = 0$ we obtain $\delta(a,b) = a^{-2}$, so the Haar measure is $\frac{dxdy}{y^2}$. To compute the modulus, let $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & bx+y \\ 0 & 1 \end{bmatrix}$. Then $\frac{|a|dxdy}{a^2x^2} = \delta(g)\frac{dxdy}{x^2}$, hence $\delta\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = |a|^{-1}$. This shows that G is not unimodular. In particular, for $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ we have

$$\int_G \phi(h)dh = \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \phi(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix})\frac{dxdy}{x^2}; \quad \int_G \phi(gh)dh = \int_G f(h)dh; \quad \int_G \phi(hg)dh = |a|\int_G \phi(h)dh$$

2.2.1. Let B_n the upper triangular subgroup of $GL(n,\mathbb{R})$. Then

$$dg = \frac{\prod_{1\le i\le j\le n} dx_{ij}}{\prod_{k=1}^n x_{kk}^{n-k+1}} = \det(g)^{-n-1}\frac{\prod dx_{ij}}{\prod x_{kk}^k}, \quad \delta_{B_n}(g) = x_{11}^{n-1}x_{22}^{3-n}\dots x_{nn}^{n-1} = \det(g)^{-n-1}\prod_{k=1}^n x_{kk}^{2k}$$

3. Compact groups: Peter-Weyl Theorem

From now on G is a **compact** Lie group. Since the Haar measure is finite on G, we can normalize it so that $\mathrm{vol}(G) = \int_G dg = 1$.

Example: if G is finite, $\int_G \phi(x)\,dx = \frac{1}{|G|}\sum_{g\in G}\phi(g)$. For $G = \mathbb{S}^1$ then $\int_{\mathbb{S}^1}\phi(g)\,dg = \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{i\theta})d\theta$.

Recall: $\widehat{G}$ equivalence classes of irreducible (finite dimensional) representations of G. Also, we denote by G^* the set of (continuous) groups homomorphisms (group characters) $\chi : G \to \mathbb{C}^\times$.

Let $L^2(G)$ the (infinite dimensional) Hilbert space of square integrable functions ϕ with $\int_G |\phi(g)|^2dg < +\infty$. The inner product is

$$< \phi, \psi >= \int_G \phi(g)\overline{\psi}(g)\,dg, \quad \phi,\psi \in L^2(G)$$

The Peter-Weyl theorem is the result, just as in the case of finite groups, that the matrix coefficients coming from various irreducible representations of G form a Hilbert basis for $L^2(G)$. The proof is very similar to the finite groups case, so we will only comment on the differences.

3.1. Theorem.

a) [Finite dimensionality] An irreducible representation of G is necessarily finite-dimensional.

b) [Weyl trick] Every (finite dimensional) representation (π, V) of G is unitary with respect to some inner product structure on V. In particular, every representation is completely reducible.

c) [Countable dual] The set $\widehat{G}$ is countable.

3.2. Peter Weyl Theorem.

We have an equivalence of $G \times G$ representations:

$$L^2(G) = \widehat{\oplus}_{a\in\widehat{G}} V_a^* \otimes V_a$$

Since the matrix coefficients satisfy the same orthogonality relations as in the finite groups case, an equivalent statement is the following

3.3. Theorem.

a) The set $\{d_a^{1/2}\phi_{aij} : a \in \widehat{G}, 1 \le i,j \le d_a\}$ is an orthonormal (Hilbert) basis for $L^2(G)$.

b) The set$\{\chi_a : a \in \widehat{G}\}$ is an orthonormal (Hilbert) basis for $L^2_{class}(G)$.

c) Let π a finite dimensional representation of G. Then $m_a(\pi) = \langle \chi_\pi, \chi_a \rangle$, $\forall\, a \in \widehat{G}$.

In particular, π is irreducible if and only if $\|\chi_\pi\|_{L^2(G)} = 1$.

b) Fourier inversion formula: $f(g) = \sum_{a\in\widehat{G}} d_a\,\mathrm{trace}\{\pi_a(x)^*\pi_a(f)\}, \quad \forall\, f \in C^\infty(G), \forall g \in G.$

b) Parseval identity: $\int_G |f(g)|^2dg = \sum_{a\in\widehat{G}} d_a^2\|\pi_a(f)\|_{HS}^2 = \sum_a d_a \sum_{i,j} |\pi_a(f)_{i,j}|^2, \quad \forall f \in C(G).$

3.4. **Example:** $G = S^1$**.** Here $\widehat{G} = G^* = \{\chi_n : n \in \mathbb{Z}\}$, where $\chi_n(e^{i\theta}) = e^{in\theta}$. Then

$$\chi_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta = \widehat{f}(-n)$$

and hence the Fourier inversion and the Parseval formulas are

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta}, \quad \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2$$