HOMEWORK 1

Recall: the class equation. Assume $G \times X \to X$ is an action of a finite group G on a finite set X. Then $X = \sqcup_{j=1}^{k} O_j$, where O_j are distinct G-orbits. Let $x_j \in O_j$ and $\mathrm{Stab}_G(x_j) = \{g \in G : g \cdot x_j = x_j\}$ the stabilizer of x_j. Then $\mathrm{Stab}(x_j) \leq G$ is a subgroup of G (not necessarily normal) and there is a well-defined bijective map (of sets)

$$G/\mathrm{Stab}(x_j) \to O_j, \quad g\,\mathrm{Stab}(x_j) \mapsto g \cdot x_j$$

In particular, $|O_j| = \frac{|G|}{|\mathrm{Stab}(x_j)|}$ and we have the *class equation*

$$|X| = \sum_{j=1}^{k} \frac{|G|}{|\mathrm{Stab}(x_j)|}$$

Example. If $X = G$ and the action is given by conjugation: $g \cdot x = gxg^{-1}$, the orbits are called *conjugacy classes*, and the class equation counts the sizes of conjugacy classes.

The class equation may be refined a little by counting the fixed points (orbits with just one element) aside:

$$|X| = |Fix(G)| + \sum_{j=1}^{l} \frac{|G|}{|\mathrm{Stab}(y_j)|}$$

where $Fix(G) = \{x \in X : g \cdot x = x, \forall g \in G\}$ and now stabilizers appearing in the sum are proper subgroups.

1. Part I

1.1. Assume G is a finite group and p a prime number such that $p \mid |G|$. Prove that there exists $e \neq g \in G$ such that $g^p = e$.

Hint: consider the action of $\mathbb{Z}/p\mathbb{Z}$ on the set $S := \{(x_1, \ldots, x_p) \in G^p : x_1 \ldots x_p = e\}$ by cyclic permutations and apply the class equation (counting the sizes of orbits).

1.2. Assume G is a finite group with $n = p^k Q$ elements, where p is a prime, $k \geq 1$, $Q \geq 1$ and $gcd(Q, p) = 1$, so that p^k is the highest power of p dividing n. Assume $H, K \leq G$ are two subgroups of G with p^k elements. Prove that they are conjugate, i.e. there exists $g \in G$ such that $gHg^{-1} = K$.

Hint: consider the action of H on G/K by $h \cdot gK = hgK$, study the class equation, and look for fixed points.

Note: there is a resemblance (in statement but even more in proof) of this problem with a theorem in Lie theory. The theorem states that maximal tori of a compact Lie group are conjugate.

1.3. Prove that $\mathbb{Z}/p\mathbb{Z} \times Z/q\mathbb{Z} \simeq \mathbb{Z}/pq\mathbb{Z}$, for p and q distinct primes.

1.4. Let G a finite group and χ a character, that is a group homomorphism $\chi : G \to \mathbb{C}^\times$. Prove that $|\chi(g)| = 1$, $\forall g \in G$.

1.5. Assume G is finite abelian. Denote by G^* the set of characters. Prove that G^* is a finite group with the same number of elements as G. Are G and G^* isomorphic?

Hint: Structure Theorem. Every finite abelian group G is the direct product of cyclic groups, meaning: there exist $n_1, n_2, \ldots n_k$ integer numbers (not necessarily unique- see one of the previous problems) such that $G \simeq (\mathbb{Z}/n_1\mathbb{Z}) \times \ldots (\mathbb{Z}/n_k\mathbb{Z})$.

1.6. Assume $\chi_1, \chi_2 : G \to \mathbb{C}^\times$ are two characters of the finite group G. Prove that

$$\frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2}(g) = \begin{cases} 1, & \chi_1 = \chi_2 \\ 0, & \text{otherwise} \end{cases}$$

Hint: change of variable in the summation.

Date: due Tuesday, Feb 14.

2. Part II

A **Hilbert structure** (positive definite hermitian inner product) on a finite dimensional $\mathbb{C}$-vector space V is a map

$$<,>: V \times V \to \mathbb{C}$$

satisfying the following properties:

- $< \lambda_1 v_1 + \lambda_2 v_2, w >= \lambda_1 < v_1, w > + \lambda_2 < v_2, w >, \forall v_1, v_2, w \in V, \lambda_1, \lambda_2 \in \mathbb{C}$
- $< v, w >= \overline{< w, v >}$ [hermitian]
- $< v, v >\geq 0$, for all $v \in V$, and $< v, v >= 0 \iff v = 0$ [positive definite]

Such an inner product determines a norm $\|v\| = \sqrt{< v, v >}$. Implicit properties are

- $\|\lambda v\| = |\lambda\| \cdot \|v\|, \forall \lambda \in \mathbb{C}, v \in V$.
- triangle inequality: $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$
- Schwartz inequality: $| < v, w > | \leq \|v\| \cdot \|w\|$.

Examples:

- On $\mathbb{C}^n$, $< z, w >= \sum_{k=1}^n z_k \overline{w}_k$, for $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. Then $\|z\| = \left(\sum_{k=1}^n |z_k|^2 \right)^{1/2}$.
- On $\mathbb{C}^n$: $< z, w >= \sum_{k=1}^n a_k z_k \overline{w}_k$, where $a_1, \ldots, a_n > 0$ are fixed positive numbers.
- On $\mathbb{C}^n$: $< z, w >= \sum_{k=1}^n \sum_{l=1}^n a_{kl} z_k \overline{w}_l$, where $A = (a_{kl})$ is a positive definite hermitian matrix, $A = A^*$ (conjugate transpose).

Definition. The representation (π, V) of G is *unitary* if V has a Hilbert structure $<,>$ that is invariant under G:

$$< \pi(g)v, \pi(g)w >=< v, w > \quad \forall g \in G, v, w \in V$$

2.1. Assume V is a (finite dimensional) Hilbert space and $W \leq V$ is a $\mathbb{C}$-subspace. Prove that the orthogonal complement $W^\perp := \{u \in V :< u, w >= 0 \forall\, w \in W\}$ is a a subspace of V and $V = W \oplus W^\perp$.

2.2. Assume (π, V) is a unitary representation of G and $W \leq V$ is a G-invariant subspace. Prove that $W^\perp$ is G-invariant.

2.3. Consider the following representation of R in $\mathbb{C}^2$:

$$\pi : \mathbb{R} \to GL_2(\mathbb{C}), \quad \pi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

Prove that π is *not* irreducible. Can you write $\mathbb{C}^2 = U_1 \oplus U_2$, where U_1 and U_2 are invariant subspaces?

3. Part III

Definitions. Assume (π_1, V_1) and (π_2, V_2) are two representations of a group G. A linear map $T : V_1 \to V_2$ satisfying the property:

$$T(\pi_1(g)v) = \pi_2(g)(T(v)), \quad \forall g \in G, \forall v \in V_1$$

is called a homomorphism of G-representations (or of G-modules). We refer to the above property of T as G-equivariance. The set of G-equivariant homomorphisms is a linear subspace of $\mathcal{L}(V_1, V_2)$ and we denote it by $Hom_G(V_1, V_2)$.

Two representations (π_1, V_1) and (π_2, V_2) of the same group G are said to be *equivalent* if there exists a G-equivariant linear isomorphism $T : V_1 \to V_2$. Otherwise, we say they are *inequivalent*.

3.1. Assume (π_1, V_1) and (π_2, V_2) are *inequivalent* representations of G. Prove that $Hom_G(V_1, V_2) = 0$.

Hint: analyze the kernel and the image of an element $T \in Hom_G(V_1, V_2)$, and see if they are G-invariant or not.

3.2. Assume (π, V) is an irreducible representation of G. Prove that $Hom_G(V, V) = \mathbb{C} \cdot Id_V$. (Meaning the only endomorphisms of V are scalar multiples of the identity map).