HOMEWORK 10

0.1. Consider the following elements in $G = SU(3)$: $w_{jk} = E_{jk} - E_{kj} + \sum_{l \neq j,k} E_{ll}$, for $j \neq k$. For example

$$w_{31} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Prove that for any representation (π, V) of $SU(3)$

$$\pi(w_{jk}) : V(\lambda) \to V(s_{jk}\lambda)$$

where $s_{jk} : i\mathfrak{t}^* \to i\mathfrak{t}^*$ is the reflection across the line $\alpha_{jk}^{\perp}$. In particular, $V(\lambda)$ and $V(s_{jk}\lambda)$ have the same dimension.

0.2. Assume π is an irreducible representation of $SU(3)$ and $\lambda_0 = a_1 l_1 + a_2 l_2 + a_3 l_3$ is its highest weight. We have seen that the weights $\mathcal{E}(\pi)$ occur inside a (hexagonal) polygon with one of the vertices at λ_0. Prove that the other vertices occur at the weights

$$\lambda' = a_{\sigma(1)} l_1 + a_{\sigma(2)} l_2 + a_{\sigma(3)} l_3$$

where σ varies over the permutations of the set $\{1, 2, 3\}$.

0.3. Let π_{st} the standard representation of $SU(3)$ on $V = \mathbb{C}^3$ (by matrix multiplication on the left). Determine the weights of π_{st} and express them in terms of l_1, l_2, l_3.

0.4. Same question as above for the representation $\operatorname{sym}^2 \pi_{st}$ of $SU(3)$ on $\operatorname{sym}^2 V$. Argue that $\operatorname{sym}^2 \pi_{st}$ is irreducible and determine its highest weight.