HOMEWORK 2

1. Part I

1.1. Assume \((\pi, V)\) is a representation (not necessarily irreducible) of the finite group \(G\). We define the \textit{character of the representation} \(\pi\), denoted \(\chi_\pi\), by

\[
\chi_\pi(g) = \text{trace}(\pi(g))
\]

Note: \(\chi_\pi\) is not a \textit{group character} (that is \(\chi_\pi(xy) \neq \chi_\pi(x)\chi_\pi(y)\)) but simply a function \(\chi_\pi : G \to \mathbb{C}\). Hopefully the name will not produce confusion.

a) Prove that \(\chi_\pi\) is conjugation invariant, that is \(\chi_\pi(xy^{-1}) = \chi_\pi(x), \forall x, y \in G\).

b) Prove that if \(\pi_1\) and \(\pi_2\) are equivalent representations (not necessarily irreducible), then \(\chi_{\pi_1} = \chi_{\pi_2}\) as functions on \(G\).

c) Let \((\pi, C^3)\) the standard representation of \(S_3\) on \(\mathbb{C}^3\). Compute \(\|\chi_\pi\|_{L^2(S_3)}\).

d) [optional] Let \((\pi, C^n)\) the standard representation of \(S_n\) on \(\mathbb{C}^n\). Compute the character \(\chi_\pi\) and the \(L^2\) norm \(\|\chi_\pi\|_{L^2(S_n)}\).

Recall: the \(L^2\)-inner product on the linear space of functions is given by

\[
\langle f, g \rangle_{L^2(G)} = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}
\]

2. Part II

2.1. a) Consider the following actions \(l(g)\) and \(r(G)\) of \(G\) on \(L^2(G)\):

\[
l(g)f(x) = f(g^{-1}x), \quad r(g)f(x) = f(xg), \quad \forall f \in L^2(G), x \in G
\]

Verify that \((l, L^2(G))\) and \((r, L^2(G))\) are unitary representations.

b) Compute the character representations of \(l\) and \(r\).

2.2. Recall: for \(V\) a vector space over \(\mathbb{C}\), \(V^* = \mathcal{L}(V, \mathbb{C})\) is the set of \(\mathbb{C}\)-linear maps \(l : V \to \mathbb{C}\).

a) Prove that \(\dim_\mathbb{C} V = \dim_\mathbb{C} V^*\).

b) Assume \((\pi, V)\) is a representation of \(G\). For \(g \in G\), define the following operator on \(V^*\)

\[
\pi^*(g) : V^* \to V^*, \quad \pi^*(g)l(v) = l(\pi(g^{-1})v), \quad \forall l \in V^*, v \in V
\]

Prove that \((\pi^*, V^*)\) is a representation (this is called the contragredient representation of \(\pi\).

Note: in general, \(\pi\) and \(\pi^*\) are not equivalent.

c*) [optional] Give a necessary and sufficient condition for \(\pi\) and \(\pi^*\) to be equivalent representations.

3. Part III

3.1. Assume \(V, W\) are finite dimensional Hilbert spaces and \(B : V \times W \to \mathbb{C}\) is a function with the following properties (skew-linear):

\[
\begin{align*}
B(c_1v_1 + c_2v_2, w) &= c_1B(v_1, w) + c_2B(v_2, w), \quad \forall c_i \in \mathbb{C}, v_i \in V, w \in W \\
B(v, c_1w_1 + c_2w_2) &= c_1B(v, w_1) + c_2B(v, w_2), \quad \forall c_i \in \mathbb{C}, v \in V, w_i \in W
\end{align*}
\]

a) Prove that there exists a linear map \(T \in \mathcal{L}(V, W)\) such that \(B(v, w) = \langle Tv, w \rangle\).

b) Assume \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are two representations (not necessarily irreducible) of the same group \(G\). Assume \(B : V_1 \times V_2 \to \mathbb{C}\) is a skew-linear map as above, which moreover has the \(G\)-invariance property

\[
B(\pi_1(g)v_1, \pi_2(g)v_2) = B(v, w)
\]

Prove that the corresponding linear map \(T\) is \(G\)-equivariant, that is \(T \in Hom_G(V_1, V_2)\).
c) Assume \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are 
\textit{inequivalent} irreducible representation, and let \(v_1, w_1 \in V_1\) and 
\(v_2, w_2 \in V_2\). Prove that the corresponding matrix coefficients 
\[ c_{v_1 w_1}(g) = \langle \pi_1(g)v_1, w_1 \rangle, \quad c_{v_2 w_2}(g) = \langle \pi_2(g)v_2, w_2 \rangle \]
are orthogonal in \(L^2(G)\), that is 
\[ \langle c_{v_1 w_1}, c_{v_2 w_2} \rangle_{L^2(G)} = 0 \]

3.2. \textbf{[optional].} Let \((\pi, V)\) be an irreducible representation of the finite group \(G\). Assume \(\langle, \rangle_1\) and 
\(\langle, \rangle_2\) are two \(G\)-invariant Hilbert structures, i.e. \(\pi\) is unitary with respect to both \(\langle, \rangle_1\) and \(\langle, \rangle_2\). Use 
Schur's lemma to prove that there exists \(c > 0\) a positive real number such that 
\[ \langle x, y \rangle_1 = c \langle x, y \rangle_2, \quad \forall x, y \in V. \] (In other words, the unitary structure on an irreducible representation is unique up to scalars).

4. \textbf{Part IV}

4.1. Assume \(A \in GL_N(\mathbb{C})\) has the property that \(A^k = I_N\), for some \(k \geq 1\) integer. Use group theoretic 
arguments (Weyl unitary trick for a certain group) to prove that \(A\) is diagonalizable.