HOMEWORK 2

1. Part I

1.1. Assume (π, V) is a representation (not necessarily irreducible) of the finite group G. We define the *character of the representation* π, denoted χ_π, by

$$\chi_\pi(g) = \text{trace}(\pi(g))$$

Note: χ_π is not a *group character* (that is $\chi_\pi(xy) \neq \chi_\pi(x)\chi_\pi(y)$) but simply a function $\chi_\pi : G \to \mathbb{C}$. Hopefully the name will not produce confusion.

a) Prove that χ_π is conjugation invariant, that is $\chi_\pi(xyx^{-1}) = \chi_\pi(x)$, $\forall x, y \in G$.

b) Prove that if π_1 and π_2 are equivalent representations (not necessarily irreducible), then $\chi_{\pi_1} = \chi_{\pi_2}$ as functions on G.

c) Let $(\pi, \mathbb{C}^3)$ the standard representation of S_3 on $\mathbb{C}^3$. Compute $\|\chi_\pi\|_{L^2(S_3)}$.

d) [optional] Let $(\pi, \mathbb{C}^n)$ the standard representation of S_n on $\mathbb{C}^n$. Compute the character χ_π and the L^2 norm $\|\chi_\pi\|_{L^2(S_n)}$.

Recall: the L^2-inner product on the linear space of functions is given by

$$< f, g >_{L^2(G)} = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$$

2. Part II

2.1. a) Consider the following actions $l(g)$ and $r(G)$ of G on $L^2(G)$:

$$l(g)f(x) = f(g^{-1}x), \quad r(g)f(x) = f(xg), \quad \forall f \in L^2(G), x \in G$$

Verify that $(l, L^2(G))$ and $(r, L^2(G))$ are unitary representations.

b) Compute the character representations of l and r.

2.2. Recall: for V a vector space over $\mathbb{C}$, $V^* = \mathcal{L}(V, \mathbb{C})$ is the set of $\mathbb{C}$-linear maps $l : V \to \mathbb{C}$.

a) Prove that $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V^*$.

b) Assume (π, V) is a representation of G. For $g \in G$, define the following operator on V^*

$$\pi^*(g) : V^* \to V^*, \quad \pi^*(g)l(v) = l(\pi(g^{-1})v), \quad \forall l \in V^*, v \in V$$

Prove that (π^*, V^*) is a representation (this is called the contragredient representation of π.

Note: in general, π and π^* are *not* equivalent.

c*) [optional] Give a necessary and sufficient condition for π and π^* to be equivalent representations.

3. Part III

3.1. Assume V, W are finite dimensional Hilbert spaces and $B : V \times W \to \mathbb{C}$ is a function with the following properties (skew-linear):

$$\begin{cases} B(c_1v_1 + c_2v_2, w) = c_1B(v_1, w) + c_2B(v_2, w), & \forall c_i \in \mathbb{C}, v_i \in V, w \in W \\ B(v, c_1w_1 + c_2w_2) = \bar{c}_1B(v, w_1) + \bar{c}_2B(v, w_2), & \forall c_i \in \mathbb{C}, v \in V, w_i \in W \end{cases}$$

a) Prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that $B(v, w) =< Tv, w >$.

b) Assume (π_1, V_1) and (π_2, V_2) are two representations (not necessarily irreducible) of the same group G. Assume $B : V_1 \times V_2 \to \mathbb{C}$ is a skew-linear map as above, which moreover has the G-invariance property

$$B(\pi_1(g)v_1, \pi_1(g)v_2) = B(v, w)$$

Prove that the corresponding linear map T is G-equivariant, that is $T \in Hom_G(V_1, V_2)$.

c) Assume (π_1, V_1) and (π_2, V_2) are *inequivalent* irreducible representation, and let $v_1, w_1 \in V_1$ and $v_2, w_2 \in V_2$. Prove that the corresponding matrix coefficients

$$c_{v_1w_1}(g) = < \pi_1(g)v_1, w_1 >, \quad c_{v_2w_2}(g) = < \pi_2(g)v_2, w_2 >$$

are orthogonal in $L^2(G)$, that is

$$\left\langle c_{v_1w_1}, c_{v_2w_2} \right\rangle_{L^2(G)} = 0$$

3.2. **[optional].** Let (π, V) be an irreducible representation of the finite group G. Assume $<,>_1$ and $<,>_2$ are two G-invariant Hilbert structures, i.e. π is unitary with respect to both $<,>_1$ and $<,>_2$. Use Schur's lemma to prove that there exists $c > 0$ a positive real number such that $< x, y >_1 = c < x, y >_2$, $\forall x, y \in V$. (In other words, the unitary structure on an irreducible representation is unique up to scalars).

4. Part IV

4.1. Assume $A \in GL_N(\mathbb{C})$ has the property that $A^k = I_N$, for some $k \geq 1$ integer. Use group theoretic arguments (Weyl unitary trick for a certain group) to prove that A is diagonalizable.