MATH 423: HOMEWORK 3

1. Part I

1.1. Let (π, V) an arbitrary representation and define $V^G = \{v \in V : \pi(g)v = v, \forall g \in G\}$ the linear subspace of G-invariant vectors. Prove that $\dim V^G = \frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)$.

1.2. Assume (π, V) is a representation of G and (π^*, V^*) the contragredient representation (defined in the previous homework).

a) Prove that $\chi_{\pi^*} = \overline{\chi}_\pi$.

b) Conclude that $\pi \simeq \pi^*$ (in this case π is *self-dual*) if and only χ_π is real-valued.

1.3. For (π, V) a representation of G and $f : G \to \mathbb{C}$ a function on G, we define the linear operator

$$\pi(f) : V \to V, \quad \pi(f) = \frac{1}{|G|}\sum_{g\in G} f(g)\pi(g)$$

In other words, for $v \in V$, $\pi(f)(v) = \frac{1}{|G|}\sum_{x\in G} f(g)\pi(g)(v)$. Prove that

$$\operatorname{trace}\pi(f) = < f, \chi_{\pi^*} >$$

where π^* is the contragredient representation.

2. Part II: class functions

Definition: a function $f : G \to \mathbb{C}$ is called a *class function* if it invariant under conjugation, that is, if $f(xyx^{-1}) = f(x)$, $\forall x, y \in G$. We denote by $L^2_{class}(G) \subset L^2(G)$ the linear subspace of class functions.

We have seen already that characters of representations are class functions. In this homework you will have to prove that they actually form a basis for L^2_{class}.

2.1. Assume $f \in L^2(G)_{class}$ is a class function (invariant under conjugation). Let $a = (\pi_a, V_a) \in \widehat{G}$ an irreducible representation. Prove that $\pi_a(f) = \lambda \cdot I_a$, where $\lambda = d_a^{-1} < f, \chi_{a^*} >_{L^2}$ and I_a is the identity operator on V_a.

Hint: use Schur's lemma, take the trace.

2.2. Prove that $\{\chi_a : a \in \widehat{G}\}$ is an ON basis for $L^2_{class}(G)$.

Directions: check first that this is an ON family. To check that it spans, assume there exists a class function f such that $< f, \chi_a >= a$, $\forall a$ (the same idea that we used in class). Then use the previous exercise and the Fourier inversion formula from class to conclude that $f = 0$.

2.3. Use the previous problem to deduce the identity $|\widehat{G}| = |Conj(G)|$, where $Conj(G)$ is the set of conjugacy classes of G.

3. Part III: finding representations

3.1. Assume G is a finite, non-abelian group. Prove that G has an irreducible representation in dimension strictly greater than one.

Note: this complements a theorem we encounter earlier, namely that abelian groups do not have irreducible representations in dimension greater than one.

3.2. Let $\sigma_1 = (134)(2657)(89)$ and $\sigma_2 = (29)(5374)(61)$ in S_{10}. Find $g \in S_{10}$ such that $g\sigma_1 g^{-1} = \sigma_2$.

3.3. Determine the number of irreducible representations of S_4 and the dimensions in which they occur.

Hint: use the previous exercise as a hint to understand how many conjugacy classes there are in S_4.