HOMEWORK 4

1. Part I

1.1. For $a \in \widehat{G}$, define $\phi_a : G \to \mathbb{C}$ by $\phi_a(x) = d_a \overline{\chi}_a(x)$. For (π, V) an arbitrary representation of G, prove that $\pi(\phi_a)$ is the orthogonal projection onto the isotypic component $V(a)$.

1.2. Explain (briefly) the meaning of the matrices P_1, P_2, P_3 constructed on p. 29-33 of the textbook.

1.3. Decompose $\wedge^2 U$ into irreducible components, where $U \leq \mathbb{C}^3$ is the standard irreducible representation of S_3 ($\dim U = 2$).

1.4. Exercise 1F/p. 41 (textbook).

1.5. In the previous homework we saw that a group G has two different actions on the linear space $L^2(G)$, the left-regular and the right-regular action defined by

$$L(g)f(x) = f(g^{-1}x), \quad R(g)f(x) = f(xg), \quad g \in G, f \in L^2(G), x \in G$$

a) Prove that L and R are equivalent representations by computing the trace.

b) Construct an explicit linear isomorphism $T : L^2(G) \to L^2(G)$ which intertwines the two actions, in other words

$$T(L(g)f) = R(g)(Tf), \quad \forall g \in G, f \in L^2(G)$$

c) For $a \in \widehat{G}$, compute the multiplicity of a in the left-regular representation.

1.6. a) Prove that $L^2(G)$ carries a $G \times G$ representation, denoted $L \times R$.

b) Check that $\Phi_a : V_a^* \otimes V_a \to L^2(G)$ given by $\Phi_a(u^* \otimes v)(x) = u^*(\pi_a(g)v)$ is $G \times G$-equivariant and that it is norm preserving, that is $\|\Phi_a(A)\|_{L^2(G)} = \|A\|_{HS}$, $\forall A \in V_a^* \otimes V_a$.

c) Note that this means $L^2(G) = \bigoplus_{a \in \widehat{G}} V_a^* \otimes V_a$ as $G \times G$-representations.

2. Optional exercises

2.1. A representation (π, V) is called faithful if the map $\pi : G \to GL(V)$ is injective. Such a faithful representation identifies G with a subgroup of $GL(V)$. The following statement is a theorem of Burnside: Assume (π, V) is a faithful representation of G. Prove that $\forall a \in \widehat{G}$, $\exists N \geq 1$ such that $a \leq \otimes^N \pi$. In other words: given a faithful representations, each irreducible representation occur as an irreducible component of some higher tensor product.

Hint: fix $a \in \widehat{G}$ and consider the power series $F(t) = \sum_{n=0}^{\infty} m_a(\otimes^n \pi) t^n$ in the indeterminate t. What you need to show is that F is not identically 0. You can bring $F(t)$ to a simpler form using the orthogonality relations.

Date: due Monday, March 6.