MATH 423: HOMEWORK 6

1. Part I

In this section: $G = SL(2, \mathbb{R})$ and $\mathfrak{g} = sl(2, \mathbb{R})$. An element $X \in \mathfrak{g}$ is called *regular* if and only if its eigenvalues (as matrix) are distinct. The set of regular elements is $\mathfrak{g}_{reg}$.

Notation: $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

1.1. Let $X \in \mathfrak{g}_{reg}$. Prove that there exists $g \in G$ such that either $\mathrm{Ad}(g)X \in \mathbb{R}H$ or $\mathrm{Ad}(g)X \in \mathbb{R}T$.

1.2. Assume $\mathfrak{h} \leq \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ (over $\mathbb{R}$) of dimension two, $\dim_{\mathbb{R}} \mathfrak{h} = 2$. Prove that there exists $X, Y \in \mathfrak{h}$ such that $[X, Y] = 2Y$.

1.3. Determine all Lie subalgebras of $sl(2, \mathbb{R})$ of dimension ≤ 2. Prove that each of them corresponds to a **closed** Lie subgroup of $SL(2, \mathbb{R})$.

2. Part II

In this section: $G = SL(4, \mathbb{R})$ and $\mathfrak{g} = sl(4, \mathbb{R})$. Consider the following elements in $\mathfrak{g}$:

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

2.1. Let $\mathfrak{h} = \mathbb{R}T_1 \oplus \mathbb{R}T_2$. Prove that $\mathfrak{h}$ is the Lie algebra of a Lie subgroup H of G such that $H \simeq \mathbb{T}^2$ (the two-dimensional torus).

2.2. Let $U = 2\pi(T_1 + \sqrt{2}\,T_2)$. Prove that the one-parameter subgroup $\exp(tU)$ is *not* a closed subgroup of $SL(2, \mathbb{R})$.

3. Part III

3.1. Determine the real Lie algebras $su(2)$ of $so(3)$, find generators and determine their bracket relations.

3.2. Prove that $su(2)$ and $so(3)$ are isomorphic Lie algebras.

4. Part IV

4.1. Assume G is a connected Lie group with Lie algebra $\mathfrak{g}$. Prove that G is abelian if and only if $\mathfrak{g}$ is abelian.