The probability law of the Brownian motion divided by its range.

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Abstract
In the present paper we deduce explicit formulas for the probability laws of the quotients $X_t/R_t$ and $m_t/R_t$, where $X_t$ is the standard Brownian motion and $m_t, M_t, R_t$ are its running minimum, maximum and range, respectively. The computation makes use of standard techniques from analytic number theory and the theory of the Hurwitz zeta function.

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1 Introduction
The connection between the Riemann zeta function and its allies (Jacobi theta function, Hurwitz zeta function) on the one hand, and the probability laws of various processes associated to the standard Brownian motion, on the other, is well established. (See [BPY01] for a comprehensive survey.) In the present paper we add two new results to this theme.

Let $X_t$ be the standard one-dimensional Brownian motion: this is a Wiener-Levy process with mean zero and covariance $\text{cov}(X_s, X_t) = s \wedge t$. We use the following notations for the max, min, and range of $X_t$: 

$$M_t = \max_{0 \leq s \leq t} X_s, \quad m_t = -\min_{0 \leq s \leq t} X_s, \quad R_t = M_t + m_t.$$ 

(1.1)

For a fixed $t > 0$, we define the following quotients:

$$\bar{X} = X_t/R_t, \quad Q = m_t/R_t.$$ 

(1.2)

The random variable $\bar{X}$ is bounded between $-1$ and $1$ and, by the scaling property of the Brownian motion, its distribution is independent of $t$. The following theorem gives an explicit formula for its probability law.

Theorem 1.1. The distribution of $\bar{X}$ is supported in the interval $[-1, 1]$, symmetric around zero and, for $0 < v < 1$,

$$P(\bar{X} \leq v) = \frac{1 + v}{2} + \frac{v^2(1 - v)}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n - v)^2} - \frac{1}{(2n + v)^2} \right].$$ 

(1.3)

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The probability law of $Q$ was given in [Cs80, eq. (2.5)]. We state it here as well and provide a new proof, which is similar to that of Theorem 1.1.

**Theorem 1.2.** [Cs80, Csáki] The distribution of $Q$ is supported in the interval $[0, 1]$, symmetric about $1/2$ and, for $0 < v < 1$,

$$P(Q \leq v) = v(1 - v) \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n - v} + \frac{1}{n + v} \right)$$

$$= 1 - v + \frac{1}{2} v(1 - v) \left( \psi\left(\frac{v}{2}\right) + \psi\left(1 - \frac{v}{2}\right) - \psi\left(\frac{1 - v}{2}\right) - \psi\left(\frac{1 + v}{2}\right) \right),$$

(1.4)

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

## 2 Proof of Theorem 1.1

It suffices to consider the case $\bar{X} = X_1/R_1$. Let $v \in (0, 1)$. We begin with the identity

$$P(\bar{X} > v) = P(R_1 < X_1/v) = \int_0^{\infty} P(R_1 < x/v, X_1 \in dx).$$

(2.1)

The joint distribution of $X_1$ and $R_1$ is given in [BS96, 1.15.8(2)]:

$$P(R_1 \leq r, X_1 \in dx) = \sum_{k=-\infty}^{\infty} [(2k + 1 - 2k(r - x)(2kr + x)]\phi(2kr + x) \cdot dx, \quad x \leq r,$$

(2.2)

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ is the probability density function of the standard normal distribution. Therefore we can re-write (2.1) as

$$P(\bar{X} > v) = \int_0^{\infty} f(x, v) dx,$$

(2.3)

where

$$f(x, v) := \sum_{k=-\infty}^{\infty} [(2k + 1 - 2kx^2(1/v - 1)(2k/v + 1)] \phi((2k/v + 1)x).$$

(2.4)

In what follows we will evaluate the integral (2.3) using the Mellin transform of $f(x, v)$.

### 2.1 The Mellin Transform

Our strategy for computing $\int_0^{\infty} f(x, v) dx$ relies on the observation that, although we cannot integrate the series (2.4) term-by-term from $0$ to $\infty$, we can integrate it against $x^s$, when $s$ is a complex number with real part $\Re(s) > 2$. In other words, we consider the Mellin transform

$$M(s) := \int_0^{\infty} f(x, v) x^s \frac{dx}{x}, \quad \Re(s) > 2.$$  

(2.5)

We prove in the Appendix (Proposition 4.2) that as a function of $x$, $f(x, v)$ is smooth and rapidly decreasing in $x$ at both ends of the interval $(0, \infty)$. This implies that $M(s)$ is in fact defined everywhere as an entire function of the complex argument $s \in \mathbb{C}$. We then go through the following steps:

- **Step 1.** Express $M(s)$ in terms of well-known Dirichlet series when $\Re(s) > 2$.
- **Step 2.** Identify $\int_0^{\infty} f(x, v) dx = M(1)$ by analytic continuation.
Step 1. When $\Re(s) > 2$, we use (2.4) to integrate term-by-term in (2.5) and obtain, through a change of variable,

$$M(s) = \sum_{k=-\infty}^{\infty} (2k+1) \int_{0}^{\infty} \phi((2k/v + 1)x) x^s \frac{dx}{x} - \frac{1}{v - 1} \sum_{k=-\infty}^{\infty} 2k(2k/v + 1) \int_{0}^{\infty} \phi((2k/v + 1)x) x^{s+1} \frac{dx}{x}$$

$$= v^s M_\phi(s) \sum_{k=-\infty}^{\infty} \frac{2k + 1}{|2k + v|^s} + v^s(v - 1) M_\phi(s + 2) \sum_{k=-\infty}^{\infty} \frac{2k(2k + v)}{|2k + v|^{s+2}}, \quad (2.6)$$

where $M_\phi(s) := \int_{0}^{\infty} \phi(x) x^s \frac{dx}{x}$ is the Mellin transform of $\phi$. This can be computed explicitly: $M_\phi(s) = \frac{1}{2\sqrt{\pi}} 2^{s/2} \Gamma(s/2)$, but all we need is that $M_\phi(s + 2) = s M_\phi(s)$ and $M_\phi(1) = 1/2$. To simplify the right-hand side of (2.6), we introduce the following Dirichlet series (cf. [Sh08, eq. 2.4] where a similar notation is used):

$$D^+(s, v) := \sum_{k=-\infty}^{\infty} \frac{1}{|2k + v|^s}, \quad D^-(s, v) := \sum_{k=-\infty}^{\infty} \frac{2k + v}{|2k + v|^{s+1}}, \quad \Re(s) > 1. \quad (2.7)$$

The following manipulation of the main term of the right-hand side of (2.6)

$$\frac{2k + 1}{|2k + v|^s} = \frac{2k + v}{|2k + v|^s} + \frac{1 - v}{|2k + v|^s} + \frac{2k(2k + v)}{|2k + v|^{s+2}} = \frac{1}{|2k + v|^s} - \frac{v(2k + v)}{2k + v},$$

allows us to express $M(s)$ in terms of $D^\pm(s, v)$ as follows:

$$M(s) = v^s M_\phi(s) D^-(s - 1, v) + v^s(v - 1) (s - 1) M_\phi(s) D^+(s, v) - s v^{s+1} (v - 1) M_\phi(s) D^-(s + 1, v), \quad \Re(s) > 2. \quad (2.8)$$

At this point we introduce the Hurwitz zeta function

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad \Re(s) > 1, \quad a \in (0, 1). \quad (2.9)$$

It was discovered by Hurwitz that, as a function of $s$, $\zeta(s, a)$ can be analytically continued to the entire complex plane, with only a simple pole at $s = 1$. Moreover, it is known that [Ap12]:

$$\lim_{s \to 1} \left[ \zeta(s, a) - \frac{1}{s - 1} \right] = -\psi(a), \quad \zeta(0, a) = \frac{1}{2} - a. \quad (2.10)$$

As an immediate consequence, it follows that both

$$D^\pm(s, v) = 2^{-s} \left\{ \zeta \left( s, \frac{v}{2} \right) \pm \zeta \left( s, 1 - \frac{v}{2} \right) \right\}$$

have meromorphic continuation to $s \in \mathbb{C}$. Moreover, we deduce from (2.10) that

$$\lim_{s \to 1} \left[ D^+(s, v) - \frac{1}{s - 1} \right] = \log(2) - \frac{1}{2} \left( \psi \left( \frac{v}{2} \right) + \psi \left( 1 - \frac{v}{2} \right) \right) \quad (2.11)$$

$$D^-(1, v) = \frac{1}{2} \left( \psi \left( 1 - \frac{v}{2} \right) - \psi \left( \frac{v}{2} \right) \right) \quad (2.12)$$

$$D^+(0, v) = 0, \quad D^-(0, v) = 1 - v \quad (2.13)$$

Step 2. We now let $s \to 1$ in the identity (2.8): we use (2.11) and (2.13) and $M_\phi(1) = 1/2$ to obtain

$$M(1) = \frac{1}{2} v(v - 1) + \frac{1}{2} v^2(1 - v) D^-(2, v) + \frac{1}{2} v D^-(0, v)$$

$$= \frac{1}{2} v^2(1 - v) D^-(2, v). \quad (2.14)$$
Since \( s = 2 \) is in the domain of convergence of the \( D^-(s,v) \),
\[
D^-(2,v) = \sum_{k=-\infty}^{\infty} \frac{2k + v}{2k + v^3} = \frac{1}{v^2} + \sum_{k=1}^{\infty} \frac{1}{(2k + v)^2} - \frac{1}{(2k - v)^2}.
\] (2.15)

We conclude that \( P(\bar{X} > v) = \int_0^\infty f(x,v)dx = M(1) \), therefore
\[
P(\bar{X} < v) = 1 - M(1) = 1 - \frac{1}{2}v^2(1-v)D^-(2,v)
\]
\[
= 1 - \frac{1}{2}v^2 - \frac{1}{2}v^2(1-v) \sum_{k=1}^{\infty} \left[ \frac{1}{(2k + v)^2} - \frac{1}{(2k - v)^2} \right],
\] (2.16)
and this finishes the proof of Theorem 1.1.

2.2 Moments
Let \( p_X(v) = \frac{d}{dv}P(\bar{X} < v) \) be the probability density function of \( \bar{X} \). It is clear from the above identity that
\[
p_X(v) = \frac{d}{dv} \left( \frac{1}{2}v^2(v-1)D^-(2,v) \right).
\] (2.17)
The numerical calculation of \( p_X(v) \) is given in the Appendix (section 4.2). We can integrate that expression numerically against test functions to obtain (the computations were done in Matlab)
\[
E[|\bar{X}|] \approx 0.4621, \quad E[\bar{X}^2] \approx 0.2813, \quad E[\bar{X}^4] \approx 0.1418.
\] (2.18)

2.3 The Taylor expansion at \( v = 0 \)
By differentiating \((n+1)\) times the identity (1.3), we obtain all the higher order derivatives of \( p_X(v) \) at \( v = 0 \):
\[
p_X^{(n)}(0) = \begin{cases} 1/2, & n = 0 \\ 0, & n = 1 \\ -(n+1)!(n-1)2^{-n}\zeta(n), & n \geq 3, \text{ odd} \\ (n+1)!n2^{-n-1}\zeta(n+1), & n \geq 2, \text{ even} \end{cases}
\] (2.19)
where \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) is the Riemann zeta function. In particular, \( p_X^{(0)}(0) = \frac{3}{2}\zeta(3) \approx 1.8031 \). This indicates that \( v = 0 \) is a local minimum for \( p_X(v) \), which explains the bimodality illustrated in Fig. 1a. The modes of the distribution of \( \bar{X} \) occur near \( \pm 0.554 \), but it seems difficult to determine them explicitly.

3 Proof of Theorem 1.2
Let \( F(y,z) := P(m_1 \leq y, M_1 \leq z) \), the joint distribution function of \( m_1 \) and \( M_1 \). Its explicit expression is given in [Fe51] and [BS96, 1.15.4]:
\[
F(y,z) = 2 \sum_{k=-\infty}^{\infty} (-1)^k \Phi((k+1)y + kz),
\] (3.1)
where \( \Phi(x) := \int_{-\infty}^{x} \phi(u)du \) is the cumulative distribution function of the standard normal distribution. For a fixed \( v \in (0,1) \), \( P(Q \leq v) = P(\frac{m_1}{m_1 + M_1} \leq v) = P(m_1 \leq \lambda M_1) \), where \( \lambda = \frac{v}{1-v} \). Therefore
\[
P(Q \leq v) = \int_{0}^{\infty} dz \int_{0}^{\lambda z} F_{yz}''(y,z) dy = \int_{0}^{\infty} F_{z}''(\lambda z, z) dz,
\] (3.2)
Finally, we can use (2.11), (2.13) and the identity of (1.4) follows from the identity 
\[ - \psi \] of (1.4) to arrive at the second identity of (1.4). The equivalence of the two separate expressions against \( z \), \( H \), \( F \), we define them and derive their main properties. For \( v \) special cases of the classical Jacobi theta functions \([Ra73, \text{Chap. 10}]\). In what follows 4.1 Theta functions

4.1 Moments

The main ingredients of the proof of Proposition 4.2 are two theta functions that are special cases of the classical Jacobi theta functions \([Ra73, \text{Chap. 10}]\). In what follows we define them and derive their main properties. For \( v \in (0,1), p \in \{0,1\} \) and \( x > 0 \), let

\[
\vartheta_p(x,v) := \sum_{k=-\infty}^{\infty} (k + v)^p e^{-\pi(k+v)^2 x}, \quad \eta_p(x,v) := \sum_{k=-\infty}^{\infty} k^p e^{2\pi ikv} e^{-\pi k^2 x}.
\]
By definition, these functions are smooth in $x > 0$, and their behavior at $\infty$ is given by
\[ \partial_\nu (x,v) = O(e^{-cx}), \quad \eta_1(x,v) = O(e^{-cx}), \quad \eta_0(x,v) = 1 + O(e^{-cx}), \quad x \to +\infty \] (4.2)
(for any $c < \pi$). The Poisson summation formula [Ra73, eq. 35.41] applied to the function $t \mapsto (t+v)^p e^{-\pi t(v+v)^2 t}$ (with $p = 0, 1$) yields the following functional equations
\[ \dot{\psi}_0(x,v) = x^{-1/2} \eta_0(1/x,v), \quad \dot{\psi}_1(x,v) = -\partial_\nu x^{-3/2} \eta_1(1/x,v). \] (4.3)
As a consequence of these identities we obtain the behavior near 0
\[ \dot{\psi}_0(x,v) = x^{-1/2} + O(e^{-c/x}), \quad \dot{\psi}_1(x,v) = O(e^{-c/x}), \quad x \to 0^+. \] (4.4)
We now turn to the analysis of the functions $f(x,v)$ and $F'_z(az,z)$, as defined in (2.4) and (3.3). For simplicity of notation, we define
\[ g(x,v) := \sum_{k=-\infty}^{\infty} [(2k+1) - 2\pi v(1-2v)k]e^{-\pi(k+v)^2 x}, \quad x > 0 \] (4.5)
so that $f(x,v) = \frac{1}{\sqrt{\pi(2\pi v)^3}} g(\frac{x}{\pi v^{3/2}})$. The behavior of $f$ as $x \to 0^+$ is deduced from that of $g$.

**Lemma 4.1.** Let $v \in (0,1), \lambda = \frac{\pi}{\sqrt{2}}$ and $x = (\frac{\pi}{8})^{1/2} \frac{1}{1-x}$. The following identities hold
\[ g(x,v) = (1-2v) \dot{\psi}_0(x,v) + 2(1-2v) x \frac{\partial_\nu \dot{\psi}_0}{\partial x}(x,v) + 2[1 + \pi v(1-2v)x] \dot{\psi}_1(x,v), \] (4.6)
\[ \left( \frac{\pi}{8} \right)^{1/2} F'_z(\lambda z, z) = \dot{\psi}_1(x, x) - v \dot{\psi}_0(x, v) + \dot{\psi}_1(x, 1-v) + \frac{v}{2} \partial_\nu \dot{\psi}_0(x, 1-v). \] (4.7)

**Proof.** By definition, $g(x,v)$ is a linear combination of the series $\sum k e^{-\pi(k+v)^2 x}$, with $j = 0,1,2$. The series corresponding to $j = 0$ and $j = 1$ are in the linear span of $\dot{\psi}_0(x,v)$ and $\dot{\psi}_1(x,v)$. As for $j = 2$, term-by-term differentiation of (4.1) yields
\[ \frac{\partial \dot{\psi}_0}{\partial x}(x,v) = -\pi \sum_{k=-\infty}^{\infty} \frac{1}{(k+v)^2} e^{-\pi(k+v)^2 x}, \]

hence the sum corresponding to $j = 2$ can be written as
\[ \sum_{k=-\infty}^{\infty} \frac{1}{k^2} e^{-\pi(k+v)^2 x} = -\frac{1}{\pi} \frac{\partial \dot{\psi}_0}{\partial x}(x,v) - v^2 \dot{\psi}_0(x,v) - 2v \dot{\psi}_1(x,v). \]
The exact identity (4.6) results from careful bookkeeping. The proof of (4.7) is similar. \hfill \Box

**Proposition 4.2.** As a function of $x$, $f(x,v)$ is smooth and rapidly decreasing as $x \to 0^+$ and $x \to +\infty$. The same holds true for $F'_z(\lambda z, z)$ as a function of $z > 0$.

**Proof.** In the case of $f(x,v)$, it is enough to prove the same statement for $g(x,v)$, when $x \to 0^+$. We differentiate the first identity from (4.3)
\[ \frac{\partial \dot{\psi}_0}{\partial x}(x,v) = -(1/2)x^{-3/2} \eta_0(1/x,v) + x^{-5/2} \partial_\nu \dot{\psi}_0(1/x,v), \]
hence $\frac{\partial \dot{\psi}_0}{\partial x}(x,v) = -(1/2)x^{-3/2} + O(e^{-c/x})$, as $x \to 0^+$ (with $c < \pi$). We use this estimate and (4.6),
\[ g(x,v) = (1-2v\frac{x}{2} + O(e^{-c/x}))[2(1-2v)x -(1/2)x^{-2} + O(e^{-c/x})] + O(e^{-c/x}), \]

as $x \to 0^+$. The leading terms cancel out conveniently and we are left with $O(e^{-c/x})$. Similarly, the proof for $F'_z(\lambda z, z)$ follows from (4.7) and (4.4). \hfill \Box
4.2 The numerical computation of \( p_X(v) \)

In this section we obtain an alternative expression for \( P(X < v) \) that is more convenient for numerical computations than the slowly convergent series (1.3). To do so, we evaluate \( D^-(2, v) \) with the aid of the Mellin transform

\[
\mathcal{M}(\vartheta_1; s) := \int_0^\infty \vartheta_1(x,v) x^s \frac{dx}{x}, \quad \Re(s) > 1 .
\]  

(4.8)

On the one hand, we can integrate (4.1) term-by-term against \( x^s \)

\[
\mathcal{M}(\vartheta_1; s) = 2^{2s-1} \pi^{-s} \Gamma(s) D^-(2s - 1, 2v), \quad \Re(s) > 1 .
\]  

(4.9)

On the other hand, the functional equation (4.3) allows us to write

\[
\mathcal{M}(\vartheta_1; s) = \pi^{-s} \sum_{k=-\infty}^{\infty} \frac{(k + v) \Gamma(s, \pi(k + v)^2)}{|k + v|^{2s}} + 2 \pi^{-s - 3/2} \sum_{k=1}^{\infty} k^{2s - 2} \sin(2\pi k v) \Gamma(3/2 - s, \pi k^2), \quad s \in \mathbb{C},
\]

(4.10)

where \( \Gamma(s, x) := \int_x^\infty e^{-t} t^{s-1} dt \) is the incomplete gamma function [AS72, 6.5.3]. Combining (4.9) and (4.10) when \( s = 3/2 \), we obtain for \( v \neq 0 \) (after replacing \( 2v \) by \( v \))

\[
D^-(2, v) = \frac{2}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \frac{\text{sgn}(k) \Gamma(3/2, \pi^2(2k + v)^2)}{(2k + v)^2} + \pi \sum_{k=1}^{\infty} k \Gamma(0, \pi k^2) \sin(\pi k v) .
\]  

(4.11)

(Here \( \text{sgn}(k) = 1 \) if \( k \geq 0 \), and \( \text{sgn}(k) = -1 \) otherwise.) Both series on the right-hand side are rapidly convergent, since \( \Gamma(3/2, x) \) and \( \Gamma(0, x) \) have exponential decay at \( \infty \). The derivative of \( D^-(2, v) \) can also be computed:

\[
\frac{\partial}{\partial v} D^-(2, v) = -\frac{4}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \frac{\text{sgn}(k) \Gamma(3/2, \pi^2(2k + v)^2)}{(2k + v)^3} - \frac{\pi}{2} \sum_{k=-\infty}^{\infty} e^{-\pi^2(2k + v)^2} \]

\[
+ \pi^2 \sum_{k=1}^{\infty} k^2 \Gamma(0, \pi k^2) \cos(\pi k v), \quad v \neq 0 .
\]  

(4.12)

The last two formulas allow us to evaluate numerically \( p_X(v) \) using (2.17), which we re-write as

\[
p_X(v) = v(3v/2 - 1) D^-(2, v) + \frac{1}{2} v^2(v - 1) \frac{\partial}{\partial v} D^-(2, v) .
\]  

(4.13)

In Matlab notation, \( \Gamma(0, x) = \text{expint}(x) \) and \( \frac{\pi}{4} \Gamma(3/2, x) = \text{gammainc}(x, 3/2, \text{'upper'}) \).

By retaining, on the right-hand side of (4.11), and (4.12), only the terms corresponding to \( |k| \leq 2 \), in the exponential series, and \( k = 1 \), in the trigonometric series, we obtain an approximation of \( p_X(t) \) within \( 10^{-6} \), uniformly in the interval \([-1, 1]\).

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(a) The distribution of $\bar{X}$.  
(b) The distribution of $Q$.

Figure 1

References


