

MORSE THEORY AND HYPERKÄHLER KIRWAN SURJECTIVITY FOR HIGGS BUNDLES

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ABSTRACT. This paper uses Morse-theoretic techniques to compute the equivariant Betti numbers of the space of semistable rank two degree zero Higgs bundles over a compact Riemann surface, a method in the spirit of Atiyah and Bott's original approach for semistable holomorphic bundles. This leads to a natural proof that the hyperkähler Kirwan map is surjective for the non-fixed determinant case.

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1. INTRODUCTION

Let E be a complex Hermitian vector bundle of rank r and degree d over a compact Riemann surface X of genus g . Let $\mathcal{A}(r, d)$ denote the space of Hermitian connections on E and let $\mathcal{A}_0(r, d)$ denote the space of traceless Hermitian connections. We use $\text{End}(E)$ to denote the bundle of endomorphisms of E and $\text{End}_0(E)$ to denote the bundle of trace-free endomorphisms of E . Let $\text{ad}(E) \subset \text{End}(E)$ and $\text{ad}_0(E) \subset \text{End}_0(E)$ denote the skew adjoint endomorphisms of E with respect to the Hermitian metric.

Let $\mathcal{B}(r, d) = \{(A, \phi) \in \mathcal{A}(r, d) \times \Omega^0(\text{End}(E) \otimes K) : d_A''\phi = 0\}$ denote the space of *Higgs bundles* of degree d and rank r over X and let $\mathcal{B}_0(r, d) = \{(A, \phi) \in \mathcal{A}_0(r, d) \times \Omega^0(\text{End}_0(E) \otimes K) : d_A''\phi = 0\}$ denote the space of *fixed determinant Higgs bundles* of degree d and rank r over X . Let \mathcal{G} denote the gauge group of E , with structure group $U(r)$ for the non-fixed determinant case, and $SU(r)$ for the fixed determinant case.

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In the following \mathcal{B} will be used to denote the space of Higgs bundles and the extra notation will be omitted if the meaning is clear from the context. \mathcal{B}^{st} (resp. \mathcal{B}^{ss}) denotes the space of *stable* (resp. *semistable*) Higgs bundles, those for which every ϕ -invariant holomorphic sub-bundle $F \subset E$ satisfies $\frac{\deg(F)}{\text{rank}(F)} < \frac{\deg(E)}{\text{rank}(E)}$ (resp. $\frac{\deg(F)}{\text{rank}(F)} \leq \frac{\deg(E)}{\text{rank}(E)}$). Let $\langle u, v \rangle = \int_X \text{tr}\{u\bar{v}\}$ be an L^2 inner product on $\Omega^0(\text{ad}(E))$, with associated norm $\|u\|^2 = \langle u, u \rangle$. The functional $\text{YMH}(A, \phi) = \|F_A + [\phi, \phi^*]\|^2$ is defined on \mathcal{B} , and let Z denote the space of Higgs bundles that minimise YMH .

Let $\mathcal{M}^{\text{Higgs}}(r, d) = Z/\mathcal{G}$ denote the *moduli space of semistable Higgs bundles* of rank r and degree d and non-fixed determinant, and let $\mathcal{M}_0^{\text{Higgs}}(r, d)$ be the corresponding space with fixed determinant.

The purpose of this paper is to use the equivariant Morse theory of the functional YMH on the space \mathcal{B} to study the topology of the moduli space of rank 2 Higgs bundles for both fixed and non-fixed determinant and both degree zero and odd degree. The methods used in this paper give the following new results for the degree zero case.

Theorem 1.1. *The equivariant Betti numbers of the space of semi-stable Higgs bundles of rank 2 and degree zero with fixed determinant over a compact Riemann surface X of genus g is given by:*

$$\begin{aligned} P_t^{\mathcal{G}}(\mathcal{B}_0^{ss}(2, 0)) &= \frac{(1+t^3)^{2g} - (1+t)^{2g}t^{2g+2}}{(1-t^2)(1-t^4)} \\ &\quad - t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g}t^{4g-4}}{4(1+t^2)} \\ &\quad + \frac{(1+t)^{2g}t^{4g-4}}{2(1-t^2)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right) \\ &\quad + \frac{1}{2}(2^{2g}-1)t^{4g-2} \left((1+t)^{2g-2} + (1-t)^{2g-2} - 2t^{2g-2} \right) \end{aligned}$$

and in the non-fixed determinant case:

$$\begin{aligned} P_t^{\mathcal{G}}(\mathcal{B}^{ss}(2, 0)) &= \frac{(1+t)^{2g}}{(1-t^2)^2(1-t^4)} \left((1+t^3)^{2g} - (1+t)^{2g}t^{2g+2} \right) \\ &\quad + \frac{(1+t)^{2g}}{1-t^2} \left(-t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g}t^{4g-4}}{4(1+t^2)} \right) \\ &\quad + \frac{(1+t)^{4g}t^{4g-4}}{2(1-t^2)^2} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right) \end{aligned}$$

The odd degree case was studied by Hitchin in [4] using the Morse theory of the functional $\frac{1}{2}\|\phi\|^2 : \mathcal{M}_0^{\text{Higgs}}(2, 1) \rightarrow \mathbb{R}$ which appears as the moment map associated to the S^1 action $e^{it} \cdot (A, \phi) = (A, e^{it}\phi)$ on the moduli space $\mathcal{M}_0^{\text{Higgs}}(2, 1)$. A corollary of the methods used here is a new proof of Hitchin's result.

Theorem 1.2 (Hitchin).

$$P_t(\mathcal{M}_0^{Higgs}(2, 1)) = P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} t^{2\mu_d} \frac{(1+t)^{2g}}{1-t^2} + \sum_{d=1}^{g-1} t^{2\mu_d} P_t(\tilde{S}^{2g-2d-1}X)$$

where $\tilde{S}^n X$ denotes the 2^{2g} -fold cover of the symmetric product $S^n X$ as described in [4] Section 7. In the non-fixed determinant case:

$$P_t(\mathcal{M}^{Higgs}(2, 1)) = (1-t^2)P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} t^{2\mu_d} (1+t)^{4g} \frac{1}{1-t^2} + \sum_{d=1}^{g-1} t^{2\mu_d} P_t(S^{2g-2d-1}X \times J(X))$$

where $\mu_d = g + 2d - 2$.

As described in [4], the space $\mathcal{A} \times \Omega^0(K \otimes \text{End}(E))$ has a hyperkähler structure, and with respect to this structure the action of the gauge group \mathcal{G} has associated moment maps $\mu_1 = F_A + [\phi, \phi^*]$ and $\mu_{\mathbb{C}} = 2id_A''\phi$. The moduli space \mathcal{M}^{Higgs} is the hyperkähler quotient of $\mathcal{A} \times \Omega^0(K \otimes \text{End}(E))$ by the action of \mathcal{G} , with associated *hyperkähler Kirwan map*:

$$\kappa_H : H_{\mathcal{G}}^*(\mathcal{A} \times \Omega^0(K \otimes \text{End}(E))) \rightarrow H_{\mathcal{G}}^*(\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))$$

induced by the inclusion $\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0) \hookrightarrow \mathcal{A} \times \Omega^0(K \otimes \text{End}(E))$.

The Morse theory techniques used to calculate the cohomology of the moduli space also lead to a natural proof of the following result.

Theorem 1.3. *The hyperkähler Kirwan map is surjective for the space of rank 2 Higgs bundles of non-fixed determinant, for both degree zero and for odd degree.*

For the case of odd degree, surjectivity has already been proven by Hausel and Thaddeus in [3] using different methods. The result proved here also applies to the previously unknown degree zero case, and the proof follows naturally from the Morse theory approach used in this paper.

In [7] it was shown that the gradient flow of YMH on the space \mathcal{B} converges to a critical point which corresponds to the graded object of the Harder-Narasimhan-Seshadri filtration of the initial conditions to the gradient flow. The functional YMH then provides a \mathcal{G} -equivariant stratification of the space \mathcal{B} , and there is a well-defined retraction of each stratum onto an associated set of critical points. This convergence result is sufficient to develop a Morse-type theory which is explained in Section 2, and used to compute the cohomology of the semistable stratum \mathcal{B}^{ss} .

In order to perform this computation, we use the result from [7], which describes the Morse index at each critical point of YMH. This index is not well-defined on each connected component of the set of critical points of YMH (the negative eigenspace of the Hessian can change dimension on a connected component), and so Morse theory cannot be used *a priori*. This paper shows that it is possible to

construct the Morse theory by hand, using the commutative diagram (10) in Section 2, and computing the cohomology groups of the stratification at each stage.

This approach for \mathcal{M}^{Higgs} is a special case of a more general method originally outlined by Kirwan, where a hyperkähler quotient $M///G$ can be studied using a two-step process. Firstly the cohomology of $\mu_{\mathbb{C}}^{-1}(0)$ is calculated using the Morse theory of $\|\mu_{\mathbb{C}}\|^2$ on M , and then the cohomology of $M///G$ can be obtained by studying the Kähler quotient of $\mu_{\mathbb{C}}^{-1}(0)$ by the group G with moment map μ_1 . In the case of $M = \mathcal{A} \times \Omega^0(K \otimes \text{End}(E))$ we have that $H_G^*(\mathcal{A} \times \Omega^0(K \otimes \text{End}(E))) = H_G^*(\mathcal{B})$. Therefore in the Higgs bundle case studied here, it only remains to study the Morse theory of $\text{YMH} = \|\mu_1\|^2$ on \mathcal{B} .

Another approach outlined by Kirwan is to use the Morse theory of the functional $\|\mu_1\|^2 + \|\mu_2\|^2 + \|\mu_3\|^2$ to study the cohomology of $M///G$. For this functional, the problem of finding all of the critical points and calculating the Morse index still remains open. Further work is planned to see whether the techniques used here could be applied to the functional

$$\|\mu_1\|^2 + \|\mu_2\|^2 + \|\mu_3\|^2 = \|F_A + [\phi, \phi^*]\|^2 + \|d_A''\phi\|^2$$

to give a different method of calculation of the cohomology of $M///G$ that could possibly be generalised to cases where $H_G^*(\mu_{\mathbb{C}}^{-1}(0))$ is not easily computable.

The formula obtained here for the equivariant cohomology of the minimum has the form

$$(1) \quad P_t^{\mathcal{G}}(B^{ss}) = P_t^{\mathcal{G}}(\mathcal{B}) - \sum_{d=0}^{\infty} t^{\lambda_d} P_t^{\mathcal{G}}(\mathcal{B}_d) + \sum_{d=1}^{g-1} t^{\lambda_d} P_t^{\mathcal{G}}(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$$

where \mathcal{B}_d denotes the d^{th} stratum of the functional YMH , λ_d is the rank of a certain bundle over the d^{th} critical set η_d representing a subset of the negative eigenspace of the Hessian of YMH at η_d , and $P_t^{\mathcal{G}}(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$ are the terms arising from the fact that the Morse index is not well-defined on the first $g-1$ critical sets.

If the space $\mathcal{B} = \mu_{\mathbb{C}}^{-1}(0)$ were smooth, the Morse index well-defined and the Morse function equivariantly perfect (for example the cases of symplectic reduction considered in [1] or [5]), then the formula for the cohomology of $M///G$ would only consist of the first two terms in (1).

By considering the space $\mathcal{A} \times \Omega^0(K \otimes \text{End}(E))$ as a cotangent bundle $T^*\mathcal{A}$, and $\mathcal{B} = \mu_{\mathbb{C}}^{-1}(0)$ as a subspace of this bundle, then on a critical set of YMH the solutions of the negative eigenvalue equation of the Hessian of $\text{YMH} = \|\mu_1\|^2$ split naturally into two components, one corresponding to the index of the restricted functional $\|\mu_1|_{\mathcal{A}}\|^2$ and one along the direction of the cotangent fibres. The dimension of the first component is well-defined over all points of the critical set (this corresponds to λ_d in the formula above), and the Atiyah-Bott lemma can be applied to the negative normal bundle defined along these directions. The dimension of the second component is not well-defined over all points of the critical set, the methods used here to deal with this show that this leads to extra terms in the Poincaré polynomial of $B\mathcal{G}$ corresponding to $P_t^{\mathcal{G}}(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$. This is further explained in Section 2 using the diagram (10).

More generally, the methods of this paper show that if M is a hyperkähler manifold and $B = \mu_{\mathbb{C}}^{-1}(0)$ has a stratification with sufficiently good local properties (such as those proved in [7] for the stratification of \mathcal{B} using the functional YMH), then the computations of Section 2 can be carried out. The key to generalising these results and doing analogous computations to Section 2 is to show that (a) the Atiyah-Bott lemma holds on some well-defined sub-bundle of the normal space to each stratum, and (b) the Poincaré polynomial $P_t^G(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$ from diagram (10) can be computed.

Future work is planned to investigate whether the methods of this paper can be used to study the cohomology of arbitrary finite-dimensional hyperkähler quotients, whether the extra terms $P_t^G(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$ from diagram (10) can be computed, and whether hyperkähler Kirwan surjectivity can be proven using these methods.

The end result of the methods used here, and the main theorem of this paper, is a calculation of the equivariant cohomology of the minimum of YMH on the space \mathcal{B} for rank 2 Higgs bundles of both fixed and non-fixed determinant, and both odd and zero degree. Unlike the situation in [1], it is difficult here to generalise the calculation to higher rank, since the extra terms in the Poincaré polynomial of the minimum corresponding to the second component of the negative directions of the Hessian are not known for rank greater than 3 (the rank 3 case was calculated by Gothen in [2] as part of the calculation of $P_t(\mathcal{M}_0^{\text{Higgs}}(3, 1))$ using the moment map associated to the S^1 action on the moduli space).

For the non-fixed determinant case, the long exact sequence obtained at each step of the Morse stratification splits into short exact sequences, thus providing a simple proof of the surjectivity of the hyperkähler Kirwan map. In the fixed determinant case, the calculation by Hitchin of $P_t(\mathcal{M}_0^{\text{Higgs}}(2, 1))$ for a compact genus 2 surface shows that $b_5(\mathcal{M}_0^{\text{Higgs}}(2, 1)) = 34$, however for genus 2, $b_5(B\mathcal{G}^{SU(2)}) = 4$, which shows that surjectivity cannot hold in general, even though the gradient flow of YMH provides a Morse-type stratification of \mathcal{B} .

This is somehow one of the key observations of this paper. It would be natural to conjecture that surjectivity of the hyperkähler Kirwan map would follow as in [5] from the inductive formula

$$(2) \quad P_t^G(\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)) = P_t^G(M) - t^{\lambda_d} \sum_d P_t^G(C_d)$$

where C_d denotes the d^{th} critical set and λ_d is the Morse index at C_d . However our formula (1) shows that at least for Higgs bundles equation (2) cannot hold, and surjectivity is dependent on the splitting of the exact sequence in the "extended" commutative diagram (10).

This paper is organised as follows. Section 2 contains the details of the Morse theory used to calculate the cohomology of the moduli space, Section 3 contains the computations of the cohomology groups of \mathcal{B}^{ss} and Section 4 uses the methods of Section 2 to prove hyperkähler Kirwan surjectivity for the non-fixed determinant case, for both $\deg(E) = 0$ and $\deg(E) = 1$.

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2. MORSE THEORY ON THE SPACE OF HIGGS BUNDLES

The purpose of this section is to calculate the \mathcal{G} -equivariant Poincaré polynomial of the space $\mathcal{B}^{ss}(2, 0)$ of semistable rank 2 degree zero Higgs bundles, for both fixed and non-fixed determinant. This is done in a natural way, using the functional YMH as a Morse function on the singular space \mathcal{B} . The key step is the commutative diagram (10), which provides a way to measure the imperfections of the Morse function YMH caused by the singularities in the space \mathcal{B} . The methods of this section are also valid for the rank 2 degree 1 case, and provide new computations of the results of [4] (fixed determinant case) and [3] (non-fixed determinant case).

Note that the general method of this calculation applies to the space of Higgs bundles $\mathcal{B}(r, d)$ of any rank and degree, however in order to generalise to higher rank a careful study needs to be made of the normal space to each Morse stratum, and the terms $H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}_{d,\varepsilon}'')$ from the diagram (10) need to be computed in terms of Bradlow spaces, a calculation that becomes difficult when $\text{rank}(E) \geq 4$.

In order to construct the Morse theory of YMH on the space \mathcal{B} , firstly recall the results of [7]

Theorem 2.1 ([7]). *The gradient flow of the functional YMH on the space \mathcal{B} with initial conditions (A_0'', ϕ_0) at time $t = 0$ converges smoothly to the graded object of the Harder-Narasimhan-Seshadri filtration associated to (A_0'', ϕ_0) . Moreover the analytic stratification induced by the gradient flow of YMH is the same as the algebraic stratification of YMH by Harder-Narasimhan type described in [3].*

Let \mathcal{B}_μ denote the subset of \mathcal{B} consisting of Higgs pairs with ϕ -invariant Harder-Narasimhan type μ , these are the strata described in Theorem 2.1. In the rank 2 case for a Higgs bundle (E, ϕ) with ϕ -invariant Harder-Narasimhan filtration $0 \subset L \subset E$ let $d = \deg L$ and set $\mathcal{B}_d = \mathcal{B}_\mu$. When E has degree 0 or 1 then the unstable strata are parametrised by the positive integers, for notation let $\mathcal{B}_0 = \mathcal{B}^{ss}$ denote the semistable stratum.

There are two consequences of Theorem 2.1 relevant to this paper. The first is that each stratum \mathcal{B}_μ \mathcal{G} -equivariantly deformation retracts onto the corresponding critical set η_μ , giving an isomorphism $H_{\mathcal{G}}^*(\mathcal{B}_\mu) \cong H_{\mathcal{G}}^*(\eta_\mu)$.

The second consequence is that the limit of the gradient flow of YMH can be described in the following way. For simplicity here we restrict attention to the rank 2 case which is relevant to this paper, for the general case see [7] for more details. Let (A_0'', ϕ_0) be the initial conditions for the gradient flow at time $t = 0$ with a ϕ -invariant Harder-Narasimhan filtration $0 \subset L \subset E$, and let $(A_\infty'', \phi_\infty)$ be the limit of the gradient flow with the ϕ -invariant Harder-Narasimhan filtration $0 \subset$

$L_\infty \subset E_\infty$. Theorem 2.1 shows that the type of the Harder-Narasimhan filtration is preserved by the gradient flow, and that the limit is isomorphic to the ϕ -invariant holomorphic splitting $E_\infty \cong L \oplus (E/L)$, with the induced Higgs structures (which is the graded object of the Harder-Narasimhan-Seshadri filtration in the rank 2 case). In particular there are isomorphisms $g_\infty : L \rightarrow L_\infty$ and $\tilde{g}_\infty : (E/L) \rightarrow (E_\infty/L_\infty)$. This result is used in this paper to show that the singularities in the normal space to each stratum retract onto the normal space of the critical set in a good way, which allows us to compute certain cohomology groups that measure these singularities.

On an unstable stratum \mathcal{B}_d the bundle E can be described as an extension of line bundles

$$0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0$$

where $\deg L_1 = d$. Using the notation of Section 7 of [1], let $\text{End}'(E)$ denote the bundle of endomorphisms of E which preserve the ϕ -invariant Harder-Narasimhan filtration, and define the quotient $\text{End}''(E)$ by the exact sequence

$$0 \rightarrow \text{End}'(E) \rightarrow \text{End}(E) \rightarrow \text{End}''(E) \rightarrow 0$$

Note that for an unstable rank 2 Higgs bundle (E, ϕ) with ϕ -invariant Harder-Narasimhan filtration $0 \subset L \subset E$, then $\text{End}''(E) \cong L^* \otimes (E/L)$.

For ε small, an ε -neighbourhood of the stratum \mathcal{B}_d at the point (A'', ϕ) consists of infinitesimal variations to the Higgs structure on E (i.e. elements $(a'', \varphi) \in \Omega^{0,1}(\text{End}(E)) \oplus \Omega^{1,0}(\text{End}(E))$ that satisfy $d''_{A+a''}(\phi + \varphi) = 0$), which are orthogonal to those variations that preserve the Harder-Narasimhan filtration, and which are orthogonal to the $\mathcal{G}^{\mathbb{C}}$ orbits. Since the bundle $\text{End}'(E)$ preserves the filtration, then the normal space consists of $(a'', \varphi) \in \Omega^{0,1}(\text{End}''(E)) \oplus \Omega^{1,0}(\text{End}''(E))$ that satisfy the following equations with respect to the induced Higgs structure (A'', ϕ) on $\text{End}''(E)$

$$(3) \quad d_A^{''*} a'' - \bar{*}[\phi, \bar{*}\varphi] = 0$$

$$(4) \quad d''_{A+a''}(\phi + \varphi) = 0$$

The second equation can be expanded to become

$$d''_A \phi + d''_A \varphi + [a'', \phi] + [a'', \varphi] = 0$$

The first term is zero since (A'', ϕ) is a Higgs bundle. When E is a rank 2 bundle then $\text{End}''(E)$ is a line bundle and so the last term is also zero, giving the linear equation

$$(5) \quad d''_A \varphi + [a'', \phi] = 0$$

Let $\mathcal{B}_{d,\varepsilon}$ denote an ε -neighbourhood of \mathcal{B}_d . Then the above argument shows that when ε is small there is a continuous map $\pi : \mathcal{B}_{d,\varepsilon} \rightarrow \mathcal{B}_d$ such that the fibre over the point $(A', \phi) \in \mathcal{B}_d$ consists of solutions to (3) and (5). A standard application of the Hodge theorem shows that

Proposition 2.2. *When E is a rank 2 bundle then for $(A'', \phi) \in \mathcal{B}_d$, the fibre $\pi^{-1}(A'', \phi)$ is isomorphic to $H^{0,1}(\text{End}''(E)) \oplus H^{1,0}(\text{End}''(E))$.*

Note that when (A'', ϕ) is a critical point of YMH then this result agrees with the calculation of the Morse index in Section 3 of [7], and similarly this shows that the normal space to each stratum is not a well-defined vector bundle over the stratum, since the dimension of $H^{1,0}(\text{End}''(E))$ depends on the holomorphic structure on $\text{End}''(E)$.

Proposition 2.2 provides a local co-ordinate system on $\mathcal{B}_{d,\varepsilon}$ as follows. Given a point $x \in \mathcal{B}_{d,\varepsilon}$ such that $\pi(x) = (A'', \phi)$ then x can be represented in the co-ordinates $(A'', \phi, \alpha, \beta)$ where $(A'', \phi) \in \mathcal{B}_d$, $\alpha \in H^{0,1}(\text{End}''(E))$ and $\beta \in H^{1,0}(\text{End}''(E))$. Define the following subsets of $\mathcal{B}_{d,\varepsilon}$

$$\begin{aligned} Z'_{d,\varepsilon} &= \{(A'', \phi, \alpha, \beta) \in \mathcal{B}_{d,\varepsilon} : \alpha = 0 = \beta\} \\ Z''_{d,\varepsilon} &= \{(A'', \phi, \alpha, \beta) \in \mathcal{B}_{d,\varepsilon} : \alpha = 0\} \\ Y_{d,\varepsilon} &= \{(A'', \phi, \alpha, \beta) \in \mathcal{B}_{d,\varepsilon} : \alpha \neq 0, \beta = 0\} \\ T_{d,\varepsilon} &= \{(A'', \phi, \alpha, \beta) \in \mathcal{B}_{d,\varepsilon} : \alpha = 0, \beta \neq 0\} \\ \mathcal{B}'_{d,\varepsilon} &= \mathcal{B}_{d,\varepsilon} \setminus Z'_{d,\varepsilon} \\ \mathcal{B}''_{d,\varepsilon} &= \mathcal{B}_{d,\varepsilon} \setminus Z''_{d,\varepsilon} \end{aligned}$$

Excision shows that

$$H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) \cong H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon} \setminus Y_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon} \setminus Y_{d,\varepsilon})$$

Let $D = \deg E$. Since $\dim H^{0,1}(\text{End}'' E)$ is well-defined on each stratum, then applying the Thom isomorphism gives

$$\begin{aligned} H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) &\cong H_{\mathcal{G}}^{*- \nu_d}(Z''_{d,\varepsilon}) \\ H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon} \setminus Y_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon} \setminus Y_{d,\varepsilon}) &\cong H_{\mathcal{G}}^{*- \nu_d}(T_{d,\varepsilon}) \end{aligned}$$

where $\nu_d = 2 \dim_{\mathbb{C}} H^{0,1}(\text{End}''(E)) = g - 1 + 2d - D$ by Riemann-Roch. Since the fibre of $Z''_{d,\varepsilon}$ over (A'', ϕ) is isomorphic to the vector space $H^{1,0}(\text{End}''(E))$, then the deformation retraction $\beta \rightarrow 0$ shows that

$$H_{\mathcal{G}}^{*- \nu_d}(Z''_{d,\varepsilon}) \cong H_{\mathcal{G}}^{*- \nu_d}(\mathcal{B}_d)$$

Now define the following subsets of $\mathcal{B}_{d,\varepsilon}$ associated to the set of critical points $\eta_d \subset \mathcal{B}_d$ of YMH

$$\begin{aligned} \eta_{d,\varepsilon} &= \{(A'', \phi, \alpha, \beta) \in \mathcal{B}_{d,\varepsilon} : (A'', \phi) \in \eta_d\} \\ \eta''_{d,\varepsilon} &= \{(A'', \phi, \alpha, \beta) \in \eta_{d,\varepsilon} : \alpha = 0, \beta \neq 0\} \end{aligned}$$

Deformation retraction along the gradient flow shows that

$$H_{\mathcal{G}}^{*- \nu_d}(\mathcal{B}_d) \cong H_{\mathcal{G}}^{*- \nu_d}(\eta_d)$$

The dimension of $H^{1,0}(\text{End}''(E))$ is preserved by the finite-time gradient flow, since this is equivalent to the action of an element $g_t \in \mathcal{G}^{\mathbb{C}}$ (where for ease of notation here we use $\mathcal{G}^{\mathbb{C}}$ to denote the complex gauge group of the bundle $\text{End}''(E)$, rather than E). Let $0 \subset L \subset E$ be the ϕ -invariant Harder-Narasimhan filtration of (E, ϕ) , and let $0 \subset L_{\infty} \subset E_{\infty}$ be the filtration of the limit $(E_{\infty}, \phi_{\infty})$

of the gradient flow with initial conditions (E, ϕ) . The results of [7] in Theorem 2.1 show that the gradient flow converges to the graded object of the Harder-Narasimhan-Seshadri filtration, and that the isomorphisms g_t converge smoothly to an isomorphism $g_\infty : L^* \otimes (E/L) \rightarrow L_\infty^* \otimes (E_\infty/L_\infty)$. Therefore the dimension of $H^{1,0}(\text{End}''(E))$ is preserved in the limit of the gradient flow. The fibres of $T_{d,\varepsilon}$ consist of non-zero elements of $H^{1,0}(\text{End}''(E))$ and so deformation retraction along the gradient flow using the gauge transformations $g_t \in \mathcal{G}^{\mathbb{C}}$ gives the following isomorphism

$$H_{\mathcal{G}}^{*- \nu d}(T_{d,\varepsilon}) \cong H_{\mathcal{G}}^{*- \nu d}(\eta''_{d,\varepsilon})$$

These results can be summarised in the following Proposition

Proposition 2.3.

$$\begin{aligned} H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) &\cong H_{\mathcal{G}}^{*- \nu d}(\eta_{d,\varepsilon}) \\ H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) &\cong H_{\mathcal{G}}^{*- \nu d}(\eta''_{d,\varepsilon}) \end{aligned}$$

For notation, let $J(X)$ denote the Jacobian of the curve X , let $S^n X$ denote the n^{th} symmetric product of X , let $\tilde{S}^n X$ denote the 2^{2g} cover of $S^n X$ described in (7.10) of [4], and let $D = \deg(E)$. In the rank 2 case, the critical sets correspond to ϕ -invariant holomorphic splittings $E = L_1 \oplus L_2$, therefore after dividing by the unitary gauge group \mathcal{G} the critical sets of YMH are $T^*J(X) \times T^*J(X)$ (non-fixed determinant case) and $T^*J(X)$ (fixed determinant case). The set $\eta''_{d,\varepsilon}$ consists of non-zero holomorphic sections of the line bundle $L_1^*L_2$, and so the same argument as p94 of [4] shows that the space of \mathcal{G} -equivalence classes is the symmetric product $S^n X$ (non-fixed determinant case) and $\tilde{S}^n X$ (fixed determinant case), where $n = 2g - 2 + D - 2d$. Taking note of the isotropy groups for the action of \mathcal{G} , we have the following lemma

Lemma 2.4. *In the non-fixed determinant case*

$$(6) \quad H_{\mathcal{G}}^*(\eta_{d,\varepsilon}) \cong H_{\mathcal{G}}^*(\eta_d) \cong H^*(J(X) \times J(X)) \otimes H^*(BU(1))^2$$

$$(7) \quad H_{\mathcal{G}}^*(\eta''_{d,\varepsilon}) \cong H^*(J(X)) \otimes H^*(S^n X) \otimes H^*(BU(1))$$

In the fixed determinant case

$$(8) \quad H_{\mathcal{G}}^*(\eta_d) \cong H^*(J(X)) \otimes H^*(BU(1))$$

$$(9) \quad H_{\mathcal{G}}^*(\eta''_d) \cong H^*(\tilde{S}^n X)$$

Denote by $X_d = \bigcup_{i=1}^d \mathcal{B}_d$ the union of the first d strata. The goal is to compute $H_{\mathcal{G}}^*(X_d)$ in terms of $H_{\mathcal{G}}^*(X_{d-1})$ for each value of d . To do this, first consider the

following commutative diagram

$$(10) \quad \begin{array}{ccccccc} & & \vdots & & & & \\ & & \downarrow \delta^{k-1} & & & & \\ \cdots & \longrightarrow & H_{\mathcal{G}}^k(X_d, X_{d-1}) & \xrightarrow{\alpha^k} & H_{\mathcal{G}}^k(X_d) & \xrightarrow{\beta^k} & H_{\mathcal{G}}^k(X_{d-1}) \xrightarrow{\gamma^k} \cdots \\ & & \downarrow \cong & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H_{\mathcal{G}}^k(\mathcal{B}_{d,\varepsilon}, \mathcal{B}'_{d,\varepsilon}) & \xrightarrow{\alpha_\varepsilon^k} & H_{\mathcal{G}}^k(\mathcal{B}_{d,\varepsilon}) & \xrightarrow{\beta_\varepsilon^k} & H_{\mathcal{G}}^k(\mathcal{B}'_{d,\varepsilon}) \xrightarrow{\gamma_\varepsilon^k} \cdots \\ & & \downarrow \zeta^k & \nearrow \xi^k & \downarrow \omega^k & & \\ & & H_{\mathcal{G}}^k(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) & \xrightarrow{\sim e} & H_{\mathcal{G}}^k(\eta_d) & & \\ & & \downarrow \lambda^k & & & & \\ & & H_{\mathcal{G}}^k(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) & & & & \\ & & \downarrow \delta^k & & & & \\ & & \vdots & & & & \end{array}$$

where excision is used to identify $H_{\mathcal{G}}^k(\mathcal{B}_{d,\varepsilon}, \mathcal{B}'_{d,\varepsilon}) \cong H_{\mathcal{G}}^k(X_d, X_{d-1})$ and the two horizontal exact sequences are the LES of the pair (X_d, X_{d-1}) and the LES of the pair $(\mathcal{B}_{d,\varepsilon}, \mathcal{B}'_{d,\varepsilon})$. The vertical exact sequence in the diagram is the LES of the triple $(\mathcal{B}_{d,\varepsilon}, \mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$. The diagonal map ξ^k is from the LES of the pair $(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$. Applying the Atiyah-Bott lemma shows that

Lemma 2.5. *The map $\sim e : H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) \rightarrow H_{\mathcal{G}}^*(\eta_d)$ is injective and so therefore the map ξ^k is injective, since $\omega^k \circ \xi^k = \sim e$.*

From the horizontal LES of (10)

$$\frac{H_{\mathcal{G}}^*(X_{d-1})}{\text{im } \beta^k} \cong \frac{H_{\mathcal{G}}^*(X_{d-1})}{\ker \gamma^k} \cong \text{im } \gamma^k \cong \ker \alpha^{k+1}$$

and also

$$\text{im } \beta^k \cong \frac{H_{\mathcal{G}}^*(X_d)}{\ker \beta^k} \cong \frac{H_{\mathcal{G}}^*(X_d)}{\text{im } \alpha^k}$$

Therefore

$$\begin{aligned} \dim \ker \alpha^{k+1} &= \dim H_{\mathcal{G}}^*(X_{d-1}) - \dim \text{im } \beta^k \\ &= H_{\mathcal{G}}^*(X_{d-1}) - \dim H_{\mathcal{G}}^*(X_d) + \dim \text{im } \alpha^k \end{aligned}$$

Lemma 2.6. *$\ker \alpha^k \cong \ker \zeta^k$, and therefore $\dim \text{im } \alpha^k = \dim \text{im } \zeta^k$ also.*

Proof. First we prove that $\ker \alpha_\varepsilon^k = \ker \zeta^k$

$$(11) \quad \begin{aligned} u \in \ker \alpha_\varepsilon^k &\Leftrightarrow \alpha_\varepsilon^k(u) = 0 \Leftrightarrow \xi^k \circ \zeta^k(u) = 0 \\ &\Leftrightarrow \zeta^k(u) = 0 \end{aligned}$$

since ξ^k is injective. To show that $\ker \alpha^k \cong \ker \alpha_\varepsilon^k$, note that $\ker \alpha^k \subseteq \ker \alpha_\varepsilon^k$ follows immediately from the diagram (10), and it only remains to show that $\ker \alpha_\varepsilon^k \subseteq \ker \alpha^k$ to complete the proof. This is done in the following claim

Claim 2.7. $\text{im } \delta^{k-1} \subseteq \ker \alpha^k$

Consider the following commutative diagram

$$(12) \quad \begin{array}{ccccc} & & H_{\mathcal{G}}^{k-1}(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) & & \\ & \delta_*^{k-1} \swarrow & \downarrow \delta^{k-1} & \searrow c^{k-1} & \\ & & H_{\mathcal{G}}^k(\mathcal{B}_{d,\varepsilon}, \mathcal{B}'_{d,\varepsilon}) & & \\ & \cong \nearrow & \downarrow & \nwarrow & \\ H_{\mathcal{G}}^k(X_d, X_{d-1}) & \xrightarrow{\quad} & H_{\mathcal{G}}^k(X_d, \mathcal{B}'_{d,\varepsilon}) & \xrightarrow{\alpha^k} & H_{\mathcal{G}}^k(X_d, \mathcal{B}''_{d,\varepsilon}) \\ \downarrow \alpha^k & & \downarrow & & \downarrow j^k \\ & & H_{\mathcal{G}}^k(\mathcal{B}_{d,\varepsilon}) & & \\ & \nearrow i^k & \downarrow & \nwarrow & \\ H_{\mathcal{G}}^k(X_d) & \xrightarrow{\quad} & H_{\mathcal{G}}^k(X_d) & \xrightarrow{\cong} & H_{\mathcal{G}}^k(X_d) \end{array}$$

The map δ_*^{k-1} is the composition of the excision isomorphism $H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}'_{d,\varepsilon}) \rightarrow H_{\mathcal{G}}^*(X_d, X_{d-1})$ with the map δ^{k-1} . By commutativity of the diagram (12), $\alpha^k \circ \delta_*^{k-1} = j^k \circ \alpha^k \circ c^{k-1}$. However $\alpha^k \circ c^{k-1} = 0$ since these are two consecutive maps in the long exact sequence of the triple $(X_d, \mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$. Therefore $\alpha^k \circ \delta_*^{k-1} = 0$ and the claim is proved.

By exactness of the vertical exact sequence in the diagram (10), $\text{im } \delta^{k-1} = \ker \zeta^k$, and from (11) we have that $\ker \zeta^k = \ker \alpha_\varepsilon^k$. Therefore Claim 2.7 shows that $\ker \alpha_\varepsilon^k \subseteq \ker \alpha^k$, completing the proof of Lemma 2.6. \square

Lemma 2.8.

$$\dim \ker \alpha^{k+1} - \dim \text{im } \alpha^k = \dim H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) - \dim H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})$$

Proof. Using the vertical LES in diagram (10) we have

$$\ker \zeta^{k+1} \cong \text{im } \delta^k \cong \frac{H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})}{\ker \delta^k} \cong \frac{H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})}{\text{im } \lambda^k}$$

and

$$\text{im } \lambda^k \cong \frac{H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})}{\ker \lambda^k} \cong \frac{H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon})}{\text{im } \zeta^k}$$

Therefore

$$\begin{aligned} \dim \ker \zeta^{k+1} &= \dim H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) - \dim \text{im } \lambda^k \\ &= \dim H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) - \dim H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) + \dim \text{im } \zeta^k \end{aligned}$$

and so using Lemma 2.6:

$$\begin{aligned} \dim \ker \alpha^{k+1} - \dim \text{im } \alpha^k &\cong \dim \ker \zeta^{k+1} - \dim \text{im } \zeta^k \\ &\cong \dim H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) - \dim H_{\mathcal{G}}^*(\mathcal{B}'_{d,\varepsilon}, \mathcal{B}''_{d,\varepsilon}) \end{aligned}$$

completing the proof. \square

Combining these lemmas gives us the following formulas

$$\begin{aligned}
\dim H_{\mathcal{G}}^*(X_d) - \dim H_{\mathcal{G}}^*(X_{d-1}) &= \dim \operatorname{im} \beta^k + \dim \ker \beta^k \\
&\quad - \dim \operatorname{im} \gamma^k - \dim \ker \gamma^k \\
&= \dim \operatorname{im} \beta^k + \dim \operatorname{im} \alpha^k \\
&\quad - \dim \ker \alpha^{k+1} - \dim \operatorname{im} \beta^k \\
&= \dim \operatorname{im} \alpha^k - \dim \ker \alpha^{k+1} \\
&= \dim H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}, \mathcal{B}_{d,\varepsilon}'') - \dim H_{\mathcal{G}}^*(\mathcal{B}_{d,\varepsilon}', \mathcal{B}_{d,\varepsilon}'')
\end{aligned}$$

In the fixed determinant case this becomes

$$\begin{aligned}
\dim H_{\mathcal{G}}^*(X_d) - \dim H_{\mathcal{G}}^*(X_{d-1}) \\
= \dim (H^{*-2\nu_d}(J(X)) \otimes H^*(BU(1))) - \dim H^{*-2\nu_d}(\tilde{S}^{2g-2+D-2d}X)
\end{aligned}$$

and in the non-fixed determinant case

$$\begin{aligned}
\dim H_{\mathcal{G}}^*(X_d) - \dim H_{\mathcal{G}}^*(X_{d-1}) \\
= \dim (H^{*-2\nu_d}(J(X) \times J(X)) \otimes H^*(BU(1))^2) \\
- \dim (H^{*-2\nu_d}(S^{2g-2+D-2d}X \times J(X)) \otimes H^*(U(1)))
\end{aligned}$$

3. COMPUTATIONS OF THE EQUIVARIANT BETTI NUMBERS

Inductively computing $H_{\mathcal{G}}^*(X_d)$ in terms of $H_{\mathcal{G}}^*(X_{d-1})$ for each value of d , and using the calculation of $P_t(\tilde{S}^{2g-2d-1}X)$ in [4] gives a new proof of Theorem 7.6 (iv) in [4] (fixed determinant case) and the results of [3] (non-fixed determinant case)

Theorem 3.1.

$$P_t(\mathcal{M}_0^{Higgs}(2, 1)) = P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} t^{2\nu_d} \frac{(1+t)^{2g}}{1-t^2} + \sum_{d=1}^{g-1} t^{2\nu_d} P_t(\tilde{S}^{2g-2d-1}X)$$

and

$$\begin{aligned}
P_t(\mathcal{M}^{Higgs}(2, 1)) &= (1-t^2)P_t^{\mathcal{G}}(\mathcal{B}^{st}(2, 1)) \\
&= (1-t^2)P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} t^{2\nu_d} \frac{(1+t)^{4g}}{1-t^2} \\
&\quad + \sum_{d=1}^{g-1} t^{2\nu_d} P_t(S^{2g-2d-1}X \times J(X))
\end{aligned}$$

where $\nu_d = g + 2d - 2$.

Similarly, for the degree zero case, we have the following formula for the fixed determinant case

$$(13) \quad P_t^{\mathcal{G}}(\mathcal{B}_0^{ss}(2, 0)) = P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} t^{2\nu_d} \frac{(1+t)^{2g}}{1-t^2} + \sum_{d=1}^{g-1} t^{2\nu_d} P_t(\tilde{S}^{2g-2d-2} X)$$

and for the non-fixed determinant case

$$(14) \quad P_t^{\mathcal{G}}(B^{ss}(2, 0)) = P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} t^{2\nu_d} \frac{(1+t)^{4g}}{(1-t^2)^2} + \sum_{d=1}^{g-1} t^{2\nu_d} P_t(S^{2g-2d-2} X) \frac{(1+t)^{2g}}{1-t^2}$$

where $\nu_d = g + 2d - 1$. In [4] Section 7, an explicit formula is given for the sum $\sum_{d=1}^{g-1} t^{2\nu_d} P_t(\tilde{S}^{2g-2d-1} X)$ for $\nu_d = g + 2d - 2$, corresponding to the case where $\deg(E) = 1$. The purpose of this section is to use equations (13) and (14) together with the techniques of [4] to give an explicit formula for the equivariant Poincaré polynomial of the spaces $B_0^{ss}(2, 0)$ and $B^{ss}(2, 0)$. This formula is contained in the following theorem.

Theorem 3.2. (a) *In the fixed determinant case:*

$$\begin{aligned} P_t^{\mathcal{G}}(\mathcal{B}_0^{ss}(2, 0)) &= \frac{(1+t^3)^{2g} - (1+t)^{2g}t^{2g+2}}{(1-t^2)(1-t^4)} \\ &\quad - t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g}t^{4g-4}}{4(1+t^2)} \\ &\quad + \frac{(1+t)^{2g}t^{4g-4}}{2(1-t^2)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right) \\ &\quad + \frac{1}{2}(2^{2g}-1)t^{4g-2} ((1+t)^{2g-2} + (1-t)^{2g-2} - 2t^{2g-2}) \end{aligned}$$

(b) *In the non-fixed determinant case*

$$\begin{aligned} P_t^{\mathcal{G}}(B^{ss}(2, 0)) &= \frac{(1+t)^{2g}}{(1-t^2)^2(1-t^4)} ((1+t^3)^{2g} - (1+t)^{2g}t^{2g+2}) \\ &\quad + \frac{(1+t)^{2g}}{1-t^2} \left(-t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g}t^{4g-4}}{4(1+t^2)} \right) \\ &\quad + \frac{(1+t)^{4g}t^{4g-4}}{2(1-t^2)^2} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right) \end{aligned}$$

Proof. Firstly recall that for the rank 2 fixed determinant case

$$(15) \quad P_t(B\mathcal{G}) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)}$$

and for the non-fixed determinant case

$$(16) \quad P_t(BG) = \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)}$$

Note that using the results from equation (7.13) in [4], the last term in (13) is given by

$$(17) \quad \sum_{d=1}^{g-1} t^{2\nu_d} P_t(\tilde{S}^{2g-2d-2} X) = \sum_{d=1}^{g-1} t^{2(g+2d-1)} P_t(S^{2g-2d-2} X) \\ + (2^{2g} - 1) \sum_{d=1}^{g-1} \binom{2g-2}{2g-2d-2} t^{4g+2d-4}$$

Using the binomial theorem, the second term is

$$(18) \quad \frac{1}{2}(2^{2g} - 1)t^{4g-2} ((1+t)^{2g-2} + (1-t)^{2g-2} - 2t^{2g-2})$$

The first term is calculated in the following lemma

Lemma 3.3.

$$(19) \quad \sum_{d=1}^{g-1} t^{2(g+2d-1)} P_t(S^{2g-2d-2} X) = -t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g}t^{4g-4}}{4(1+t^2)} \\ - \frac{(t+1)^{2g}t^{4g-4}}{2(t^2-1)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right)$$

Part (b) of Theorem 3.2 immediately follows from equations (14), (16) and (19). Part (a) follows from combining equations (13), (15), (17) and (18) and (19).

Proof of Lemma. We follow the technique described in Section 7 of [4]. This uses the formula of [6] which states that $P_t(S^{2g-2d-2} X)$ is the co-efficient of $x^{2g-2d-2}$ in $\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}$, or equivalently the co-efficient of x^{2g} in $\frac{x^{2d+2}(1+xt)^{2g}}{(1-x)(1-xt^2)}$. Therefore the sum

$$\sum_{d=1}^{g-1} t^{2(g+2d-1)} P_t(S^{2g-2d-2} X)$$

is the co-efficient of x^{2g} in

$$(20) \quad \sum_{d=1}^{g-1} t^{2(g+2d-1)} x^{2d+2} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}$$

which is equal to the co-efficient of x^{2g} in the following infinite sum

$$\sum_{d=1}^{\infty} t^{2(g+2d-1)} x^{2d+2} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}$$

The sum above is equal to

$$\begin{aligned} \sum_{d=1}^{\infty} t^{2(g+2d-1)} x^{2d+2} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} &= t^{2g+2} x^4 \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \sum_{d=1}^{\infty} (xt^2)^{2(d-1)} \\ &= \frac{t^{2g+2} x^4 (1+xt)^{2g}}{(1-x)(1-xt^2)(1-x^2 t^4)} \end{aligned}$$

Therefore the co-efficient of x^{2g} in the above sum is equal to the residue at $x = 0$ of the function

$$(21) \quad f(x) = \frac{(1+xt)^{2g} t^{2g+2}}{(1-x)(1-xt^2)^2(1+xt^2)} \cdot \frac{1}{x^{2g-3}}$$

As in [4], this residue can be computed in terms of the residues at the simple poles $x = 1$ and $x = -t^{-2}$, the residue at the double pole $x = t^2$ and the integral of $f(x)$ around a contour containing all of the poles. In this case the same methods can be used to compute the residues. However, unlike the situation in [4], the contour integral is not asymptotically zero as the contour approaches the circle at infinity, so this must be computed here as well. To compute the integral, let C_r be the circle of radius r in the complex plane where $r > 1$ and $r > t^{-2}$ (ie the disk inside C_r contains all the poles of $f(x)$). Then for $|x| = r$ we have the following Laurent expansion of $f(x)$ centred at $x = 0$.

$$\begin{aligned} \frac{(1+xt)^{2g} t^{2g+2} x^{3-2g}}{(1-x)(1-xt^2)^2(1+xt^2)} &= - \frac{(\frac{1}{x} + t)^{2g} t^{2g+2}}{xt^6 (1 - \frac{1}{x}) (1 - \frac{1}{xt^2})^2 (1 + \frac{1}{xt^2})} \\ &= - \frac{1}{xt^4} \left(\frac{t}{x} + t^2 \right)^{2g} \left(1 + \frac{1}{x} + \dots \right) \\ &\quad \times \left(1 + \frac{1}{xt^2} + \dots \right)^2 \left(1 - \frac{1}{xt^2} + \dots \right) \\ &= - \frac{t^{4g-4}}{x} + \text{terms of order } x^{-n} \text{ where } n > 1 \end{aligned}$$

This series expansion is uniformly convergent on the annulus $\{x : r - \varepsilon < x < r + \varepsilon\}$ for $r > 1$, $r > t^{-2}$ and ε small enough so that the closure of the annulus doesn't contain any of the poles of $f(x)$. As $r \rightarrow \infty$ the series asymptotically approaches $-\frac{t^{4g-4}}{x}$, and so the integral approaches

$$(22) \quad \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C_r} \frac{(1+xt)^{2g} t^{2g+2} x^{3-2g}}{(1-x)(1-xt^2)^2(1+xt^2)} dx = -t^{4g-4}$$

The residues of $f(x)$ at $x = 1$, $x = -t^{-2}$ and $x = t^2$ are similar to the results obtained in [4]. At the simple pole $x = 1$

$$(23) \quad Res_{x=1} f(x) = - \frac{t^{2g+2} (1+t)^{2g}}{(1-t^2)(1-t^4)}$$

At the simple pole $x = -t^{-2}$

$$(24) \quad \text{Res}_{x=-t^{-2}} f(x) = -\frac{(1-t)^{2g} t^{4g-4}}{4(1+t^2)}$$

and at the double pole $x = t^{-2}$

$$(25) \quad \text{Res}_{x=t^{-2}} f(x) = \frac{(t+1)^{2g} t^{4g-4}}{2(t^2-1)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right)$$

Combining (22), (23), (24) and (25) we have

$$(26) \quad \sum_{d=1}^{g-1} t^{2(g+2d-1)} P_t(S^{2g-2d-2} X) = -t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g} t^{4g-4}}{4(1+t^2)} \\ - \frac{(t+1)^{2g} t^{4g-4}}{2(t^2-1)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right)$$

thus completing the proof of the lemma, and therefore Theorem 3.2 also. \square

4. RELATIONSHIP TO HYPERKÄHLER KIRWAN SURJECTIVITY

For simplicity of notation, throughout this section let $n = 2g - 2 + D - 2d$ where $D = \deg(E)$ and d is the index of the stratum \mathcal{B}_d . In this section we use the Morse theory of the previous sections to provide a proof of the following result for the non-fixed determinant case.

Theorem 4.1. *The spaces $\mathcal{M}^{\text{Higgs}}(2, 1)$ and $\mathcal{M}^{\text{Higgs}}(2, 0)$ are hyperkähler quotients $T^*A//\mathcal{G}$ for which the hyperkähler Kirwan map*

$$\kappa_H : H_{\mathcal{G}}^*(T^*A) \rightarrow H_{\mathcal{G}}^*(\mathcal{B}^{ss})$$

is surjective.

As mentioned in the introduction, for the space $\mathcal{M}^{\text{Higgs}}(2, 1)$ this result has already been proven in [3], however the methods used in this paper also apply to the space $\mathcal{M}^{\text{Higgs}}(2, 0)$ which is singular and so does not fit into the framework of [3].

The proof of Theorem 4.1 reduces to showing that the long exact sequence

$$\cdots \rightarrow H_{\mathcal{G}}^*(X_d, X_{d-1}) \rightarrow H_{\mathcal{G}}^*(X_d) \rightarrow H_{\mathcal{G}}^*(X_{d-1}) \rightarrow \cdots$$

of diagram (10) splits, and hence the map $\beta^* : H_{\mathcal{G}}^*(X_d) \rightarrow H_{\mathcal{G}}^*(X_{d-1})$ is surjective for each positive integer d . Lemma 2.6 shows that this long exact sequence splits iff the vertical long exact sequence of diagram (10) splits, and Proposition 2.3 shows that it is sufficient to prove that the map $\lambda^* : H_{\mathcal{G}}^*(\eta_{d,\varepsilon}) \rightarrow H_{\mathcal{G}}^*(\eta''_{d,\varepsilon})$ is surjective. The following lemma provides a simpler description of the map λ^* .

Lemma 4.2. *Let $D = \deg(E)$. The map λ^* restricts to a map*

$$\lambda_r^* : H^{*-2\mu_d}(J(X)) \otimes H^*(BU(1)) \rightarrow H^*(S^n X)$$

and λ^ is surjective iff λ_r^* is surjective. The map λ_r^* is induced by the Abel-Jacobi map $S^n X \rightarrow J(X)$.*

Proof. Recall that for the critical set η_μ , the following decomposition of the equivariant cohomology holds

$$(27) \quad H_{\mathcal{G}}^*(\eta_\mu) \cong H_{\mathcal{G}_{diag}}^*(\eta_\mu^*) \cong H_{G_{diag}}^*(\tilde{\eta}_\mu^*)$$

where \mathcal{G}_{diag} is the subgroup of gauge transformations which are diagonal with respect to the Harder-Narasimhan filtration, η_μ^* refers to the subset of critical points that split with respect to a fixed filtration, G_{diag} is the subgroup of constant gauge transformations that are diagonal with respect to the same fixed filtration, and $\tilde{\eta}_\mu^*$ is the fibre of $\eta_\mu^* \cong \mathcal{G}_{diag} \times_{G_{diag}} \eta_\mu^*$. In the rank 2 case, the group G_{diag} is simply the torus $T = U(1) \times U(1)$ and we can define (using the local co-ordinates on $\mathcal{B}_{d,\varepsilon}$ from the previous section)

$$(28) \quad \tilde{\eta}'_{d,\varepsilon} = \{(A'', \phi, \alpha, \beta) : (A'', \phi) \in \tilde{\eta}_d^*, \alpha = 0\}$$

$$(29) \quad \tilde{\eta}''_{d,\varepsilon} = \{(A'', \phi, \alpha, \beta) : (A'', \phi) \in \tilde{\eta}_d^*, \alpha = 0, \beta \neq 0\}$$

The map λ^* is induced by the inclusion $\tilde{\eta}''_{d,\varepsilon} \hookrightarrow \tilde{\eta}'_{d,\varepsilon}$ and so the map λ^* becomes

$$\lambda^* : H_T^*(\tilde{\eta}'_{d,\varepsilon}) \rightarrow H_T^*(\tilde{\eta}''_{d,\varepsilon})$$

Let T' be the quotient of T by the subgroup of constant multiples of the identity. Since the constant multiples of the identity fix all points in $\tilde{\eta}'_{d,\varepsilon}$ and $\tilde{\eta}''_{d,\varepsilon}$ then $H_T^*(\tilde{\eta}'_{d,\varepsilon}) \cong H_{T'}^*(\tilde{\eta}'_{d,\varepsilon}) \otimes H^*(BU(1))$ and $H_T^*(\tilde{\eta}''_{d,\varepsilon}) \cong H_{T'}^*(\tilde{\eta}''_{d,\varepsilon}) \otimes H^*(BU(1))$.

Therefore the map $\lambda^* : H_{T'}^*(\tilde{\eta}'_{d,\varepsilon}) \otimes H^*(BU(1)) \rightarrow H_{T'}^*(\tilde{\eta}''_{d,\varepsilon}) \otimes H^*(BU(1))$ is the identity on the factor $H^*(BU(1))$.

Now consider co-ordinates on $\tilde{\eta}'_{d,\varepsilon}$ given by $(L_1, L_2, \phi_1, \phi_2, \varphi)$ where $L_1, L_2 \in J(X)$ are the line bundles of the holomorphic splitting $E = L_1 \oplus L_2$ and $\varphi \in H^0(L_1^* L_2 \otimes K)$. For a fixed holomorphic structure, ϕ_1 and ϕ_2 take values in a vector space, and so $\tilde{\eta}'_{d,\varepsilon}$ is homotopy equivalent to the space with co-ordinates (L_1, L_2, φ) . This fibres over the space with co-ordinates (L, φ) where $L = L_1^* L_2 \otimes K$, with fibre $J(X)$. The fibre bundle is trivial, since it can be written with global co-ordinates as follows. Denote an element of the fibre by \tilde{L} and an element of the base by (L, φ) . Then the fibring over the point (L, φ) is given by the map $\tilde{L} \mapsto (\tilde{L}, L\tilde{L} \otimes K^*, \varphi) \mapsto (L, \varphi)$. After changing co-ordinates $(L_1, L_2, \varphi) \mapsto (L_1, L_1^* L_2 \otimes K, \varphi)$ it is clear that the fibre bundle is trivial. Therefore the cohomology of the fibre bundle splits as

$$(30) \quad H_{T'}^*(\tilde{\eta}'_{d,\varepsilon}) \cong H^*(J(X)) \otimes H_{T'}^*(J(X))$$

$$(31) \quad H_{T'}^*(\tilde{\eta}''_{d,\varepsilon}) \cong H^*(J(X)) \otimes H_{T'}^*(F_n)$$

where F_n is the space $\{(L, \varphi) : L \in J(X), \varphi \in H^0(L \otimes K)\}$, where $\deg L = n = 2g - 2 + D - 2d$. Note that F_n fibres over the symmetric product $S^n X$ with fibre $U(1) \cong T'$, where T' acts trivially on the base, and freely on the fibres. The map λ^* restricts to the identity on the factor $H^*(J(X))$ in (30) and (31), and therefore it restricts to a map $H_{T'}^*(J(X)) \rightarrow H_{T'}^*(F_n)$. Now the action of T' fixes the holomorphic structures on L_1 and L_2 , and so acts trivially on the base of the fibre bundle. T' acts freely on a non-zero section $\varphi \in H^0(L_1^* L_2 \otimes K)$ and so (after applying the deformation retraction $|\varphi| \rightarrow 1$), the quotient of the space F_n is the

space of effective divisors on X , since the zeros of each $0 \neq \varphi \in H^*(L_1^*L_2 \otimes K)$ correspond to an effective divisor of degree $2g - 2 + D - 2d$. Therefore the map λ^* restricts to a map

$$(32) \quad \lambda_r^* : H^*(J(X)) \otimes H^*(BU(1)) \rightarrow H^*(S^n X)$$

which is induced by the T' -equivariant map $F_n \rightarrow J(X)$, which maps a non-zero section $\varphi \in H^0(L_1^*L_2 \otimes K)$ to the line bundle $L_1^*L_2 \otimes K$. On the quotient $F_n/T' = S^n X$ this restricts to the Abel-Jacobi map $S^n X \rightarrow J(X)$. \square

The results of [6] describe in detail the cohomology ring of the symmetric product of a curve. In particular, $H^*(S^n X)$ is generated by $2g$ generators in H^1 and one generator in H^2 , and the proof of Theorem 4.1 reduces to showing that λ_r^* maps onto these generators. From the proof of (14.1) in [6] we have the following lemma for the Abel-Jacobi map.

Lemma 4.3. η_r^* is surjective onto $H^1(S^n X)$.

Next we need the following technical lemma.

Lemma 4.4. $H^2(F_n)$ consists of products of elements of $H^1(F_n)$.

Proof. Note that after multiplying by constant real multiples of the section φ , F_n is a fibration over $J(X)$ where fibre over the point $L \in J(X)$ is the sphere S^k , where k depends on L . By Riemann-Roch, if $n \geq 2g - 2$ then $k = 2(n - g) + 1$ is independent of L .

First consider F_N for N large. Since $N \geq 2g - 2$ then F_N is an $S^{2N-2g+1}$ -bundle over $J(X)$. For N large, from the Serre spectral sequence we have maps $H^2(J(X)) \rightarrow H^2(F_N)$ and $H^1(J(X)) \rightarrow H^1(F_N)$ (since $2N - 2g + 1 > 2$ there is no contribution from $H^1(S^{2N-2g+1}) = 0$ and $H^2(S^{2N-2g+1}) = 0$). These maps are natural with respect to cup product, and since $H^2(J(X))$ consists of products of elements of $H^1(J(X))$ then $H^2(F_N)$ consists of products of elements of $H^1(F_N)$.

Now we consider F_n , which is not a fibre bundle over the Jacobian when $n < 2g - 2$ (since the dimension of the fibre may jump). For a fixed basepoint x_0 of X , consider the inclusion map $X^n \hookrightarrow X^N$ given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x_0, \dots, x_0)$$

This induces the inclusion of symmetric products $i : S^n X \hookrightarrow S^N X$, and the description of the generators of $H^*(S^N X)$ in (3.1) of [6] shows that the induced map $i^* : H^*(S^N X) \rightarrow H^*(S^n X)$ maps generators to generators and hence is surjective. Therefore the inclusion i induces the following map of fibre bundles

$$\left(\begin{array}{ccc} U(1) & \rightarrow & F_n \\ & & \downarrow \\ & & S^n X \end{array} \right) \rightarrow \left(\begin{array}{ccc} U(1) & \rightarrow & F_N \\ & & \downarrow \\ & & S^N X \end{array} \right)$$

which is the identity map $j : U(1) \rightarrow U(1)$ on the fibres.

Since $H^2(F_N)$ has no irreducible generators, then in the Serre spectral sequence for $H^*(F_N)$, the irreducible generator $p_N \in H^2(S^N X; H^0(U(1))) \cong$

$H^2(S^N X) \otimes H^0(U(1))$ must be killed by a differential (note that π_1 acts trivially on the space of components of the fibre, and hence on $H^0(U(1))$). For dimensional reasons this must be the differential $d_2^N : E_2^{0,1} \rightarrow E_2^{2,0}$ on the E_2 page. Since the map i^* is surjective, then this shows that $i^* \circ d_2^N$ maps onto p_N . Naturality of the Serre spectral sequence then shows that $d_2^n \circ j^*$ maps onto p_n , where $d_2^n : E_2^{0,1} \rightarrow E_2^{2,0}$ is a differential on the E_2 page of the Serre spectral sequence for F_n . Since j^* is an isomorphism then this shows that d_2^n maps onto the irreducible generator p_n of $H^2(S^n X; H^0(U(1)))$. \square

Proof of Theorem 4.1. Using the definition of F_n from above, note that $S^n X \simeq F_n \times_{U(1)} EU(1)$, where $U(1)$ acts by multiplication on the fibres of $EU(1) \rightarrow F_n \rightarrow S^n X$. Therefore $S^n X$ is homotopy equivalent to a fibre bundle over F_n with fibres $BU(1)$. From the Serre spectral sequence, we have the map

$$(33) \quad (H^0(F_n) \otimes H^2(BU(1))) \oplus (H^1(F_n) \otimes H^1(BU(1))) \\ \oplus (H^2(F_n) \otimes H^0(BU(1))) \rightarrow H^2(S^n X)$$

From [6], $H^2(S^n X)$ has an irreducible generator p_n . We have that $H^1(BU(1)) = 0$ and by Lemma 4.4 there are no irreducible generators of $H^2(F_n) \otimes H^0(BU(1))$. Therefore p_n is in the image of the term $H^0(F_n) \otimes H^2(BU(1)) \cong \mathbb{C}$, and therefore this term is not killed by any differential in the Serre spectral sequence for $S^n X \simeq F_n \times_{U(1)} EU(1)$.

By construction, the map η_r^* is induced by a map of fibre bundles which is an isomorphism on the base $BU(1)$

$$\left(\begin{array}{ccc} F_n & \rightarrow & F_n \times_{U(1)} EU(1) \simeq S^n X \\ & & \downarrow \\ & & BU(1) \end{array} \right) \rightarrow \left(\begin{array}{ccc} J(X) & \rightarrow & J(X) \times_{U(1)} EU(1) \\ & & \downarrow \\ & & BU(1) \end{array} \right)$$

and therefore the induced map

$$H^2(BU(1)) \otimes H^0(J(X)) \rightarrow H^2(BU(1)) \otimes H^0(F_n)$$

is an isomorphism on the E_2 page of the respective Serre spectral sequences (again, π_1 acts trivially on the space of components of the respective fibres). Therefore we have a map

$$(34) \quad H^2(BU(1)) \otimes J(X) \rightarrow H^2(J(X) \times_{U(1)} EU(1)) \rightarrow H^2(S^n X)$$

which is surjective onto the generator p_n of $H^2(S^n)$. Together with Lemma 4.3 this completes the proof. \square

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