Morse theory for rank 2 Higgs bundles

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Preprints available at

math.DG/0611113 (Analytic Results)

math.SG/0701560 (Topological Results)

Hamiltonian group actions on symplectic manifolds

Let M be a symplectic manifold (there exists a closed, non-degenerate 2-form ω) and suppose that a Lie group G acts on M preserving the symplectic form, i.e. for each $g \in G$ there exists $\varphi_g \in \mathrm{Diff}(M)$ such that $\omega_x(X,Y) = \omega_{\varphi_g(x)}(d\varphi_g(X), d\varphi_g(Y))$.

The *infinitesimal action* of G at $x \in M$ is a map $\rho_x : \mathfrak{g} \to T_xM$ given by

$$\rho_x(u) = \frac{d}{dt}\Big|_{t=0} \exp(tu) \cdot x$$

The action is *weakly Hamiltonian* if for each $u \in \mathfrak{g}$ there exists $H_u \in C^\infty(M, \mathbb{R})$ such that

$$\omega_x(\rho_x(u), \cdot) = dH_u(x)$$

The action is *Hamiltonian* if the map $\mathfrak{g} \to C^\infty(M,\mathbb{R})$ given by $u \mapsto H_u$ is a Lie algebra homomorphism.

Moment maps and symplectic reduction

For a Hamiltonian group action we obtain a *moment map* $\mu : M \to \mathfrak{g}^*$, which satisfies $\mu(g{\cdot}x) = Ad_g^*(\mu(x))$ and $\mu(x){\cdot}u = H_u(x)$ (or $d\mu_x(X){\cdot}u = \omega(\rho_x(u), X)$).

Example: $SO(3)$ acts on $T^*\mathbb{R}^3$, the moment map is the classical angular momentum.

$$g \cdot (\mathbf{x}, \mathbf{v}) = (g\mathbf{x}, g\mathbf{v})$$

$$\mu(\mathbf{x}, \mathbf{v}) = \mathbf{x} \times \mathbf{v}$$

(identify $\mathbb{R}^3$ with $\mathfrak{so}(3)^*$)

Symplectic Reduction: Obtain a new space $M/\!/_\alpha G = \mu^{-1}(\alpha)/G$, where α is a central element of $\mathfrak{g}^*$.

$M/\!/_\alpha G = \mu^{-1}(\alpha)/G$ has the structure of a symplectic manifold away from its singularities.

Yang-Mills equations over a compact Riemann surface (Atiyah & Bott)

Let X be a compact Riemann surface, and $E \to X$ a complex vector bundle of rank n and degree d with a Hermitian metric h.

Let $\mathcal{A}$ be the space of connections on E compatible with the metric, i.e.

$$dh(\xi, \sigma) = h(d_A\xi, \sigma) + h(\xi, d_A\sigma)$$

for ξ, σ sections of E.

Curvature: $F_A\xi = d_A d_A \xi$

Yang-Mills Functional:

$$\mathrm{YM}(A) = \|F_A\|^2 = \int_X \mathrm{tr}\, F_A \bar{*} F_A$$

Yang-Mills Equations: (critical point equations for YM)

$$d_A^* F_A = 0$$

Relation to moment maps and symplectic geometry

$\mathcal{A}$ has a symplectic structure (in fact Kähler), and there is a Hamiltonian action of the gauge group $\mathcal{G}$.

$$g \cdot A = g^{-1} A g + g^{-1} dg$$

Moment map is $\mu(A) = F_A$, and so $\mathrm{YM} = \|\mu\|^2$.

The symplectic reduction is

$$\mathcal{M}^{ss}(n,d) = \mu^{-1}(2\pi i d \cdot \mathrm{id}) / \mathcal{G}$$

Theorem: (Narasimhan & Seshadri, Donaldson) $\mathcal{M}^{ss}(n,d)$ is the moduli space of semistable holomorphic structures on E.

How to study this space? Morse theory!

Atiyah & Bott: YM is $\mathcal{G}$-equivariantly perfect, and we have a surjection $H^*_{\mathcal{G}}(\mathcal{A}) \to H^*_{\mathcal{G}}\left(\mu^{-1}(2\pi i d \cdot \mathrm{id})\right)$.

Moreover we can compute the equivariant Betti numbers $\dim H^k_{\mathcal{G}}\left(\mu^{-1}(2\pi i d \cdot \mathrm{id})\right)$

Cohomology of symplectic quotients

Definition: The *equivariant cohomology* of a G-space N is

$$H^*_G(N) := H^*(EG \times_G N)$$

If the action of G is free then $H^*_G(N) \cong H^*(N/G)$.

Theorem: (Kirwan) Let M be a compact symplectic manifold with a Hamiltonian G-action. The inclusion $\mu^{-1}(\alpha) \hookrightarrow M$ induces a surjective map $H^*_G(M) \to H^*_G(\mu^{-1}(\alpha))$. Moreover, one can compute the G-equivariant Betti numbers of $\mu^{-1}(\alpha)$.

In particular, if the action of G on $\mu^{-1}(\alpha)$ is free then $M/\!\!/_\alpha G$ is smooth, and one can compute the ordinary Betti numbers of $M/\!\!/_\alpha G$.

The proof uses the Morse theory of the function

$$\|\mu - \alpha\|^2 : M \to R$$

Hyperkähler Reduction

A *hyperkähler manifold* (M, g, I, J, K) is a Riemannian manifold with three Kähler structures, which satisfy the quaternionic relations $IJ = K = -JI$.

Examples: $\mathbb{R}^{4n}$, T^{4n}, K3 surface, moduli space of semistable Higgs bundles, quiver varieties

Given the action of a Lie group G on M which is Hamiltonian with respect to each complex structure we obtain three moment maps, μ_I, μ_J, μ_K.

Hyperkähler reduction is a method for constructing new hyperkähler manifolds, a quaternionic version of symplectic reduction.

For α, β, γ central in $\mathfrak{g}^*$

$$M /\!/\!/ G = \mu_I^{-1}(\alpha) \cap \mu_J^{-1}(\beta) \cap \mu_K^{-1}(\gamma)/G$$

Higgs Bundles (Hitchin)

E a Hermitian vector bundle of rank n, degree d over a compact Riemann surface X.

Consider the space of pairs $(A,\phi) \in \mathcal{A} \times \Omega^{1,0}(\mathrm{End}(E))$.

(A,ϕ) is a *Higgs bundle* if $d''_A \phi = 0$.

Moment maps for action of $\mathcal{G}$:

$$\mu_I = F_A + [\phi, \phi^*]$$
$$\mu_{\mathbb{C}} = \mu_J + i\mu_K = 2id''_A \phi$$

Theorem: (Hitchin/Simpson) The *moduli space of semistable Higgs bundles* of rank n and degree d on X is the hyperkähler quotient

$$\mathcal{M}^{Higgs}(n,d) = \left(\mu_I^{-1}(2\pi i d \cdot \mathrm{id}) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/\mathcal{G}$$

(denoted $\mathcal{M}_0^{Higgs}(n,d)$ if the determinant of E is held fixed).

Can we compute the equivariant Betti numbers?

$$\dim H_G^k\left(\mu_I^{-1}(\alpha) \cap \mu_J^{-1}(\beta) \cap \mu_K^{-1}(\gamma)\right)$$

Is there a hyperkähler Kirwan surjectivity theorem?

$$H_G^*(M) \to H_G^*\left(\mu_I^{-1}(\alpha) \cap \mu_J^{-1}(\beta) \cap \mu_K^{-1}(\gamma)\right)$$

This was conjectured in finite dimensions by Kirwan, and proved in some special cases:
Hyperpolygon spaces (Konno)
Hypertoric varieties (Konno)
Higgs bundles (special cases) (Hausel/Thaddeus) using different methods to the ones in this talk.

Can we prove this using Morse theory in the spirit of Atiyah/Bott and Kirwan's original approach?

Can we study the topology of interesting hyperkähler quotients (such as moduli spaces of semistable Higgs bundles)??

Two different possible approaches for studying hyperkähler quotients

Let G act on a hyperkähler manifold M with real moment map μ_I and complex moment map $\mu_{\mathbb{C}}$.

1. Study the G-equivariant Morse theory of the functional $\|\mu_I\|^2 + \|\mu_{\mathbb{C}}\|^2$

OR

2. Two-step process: Study the Morse theory of $\|\mu_{\mathbb{C}}\|^2$ on M, and then study the G-equivariant Morse theory of $\|\mu_I\|^2$ on the singular space $\mu_{\mathbb{C}}^{-1}(0)$.

Is the induced map

$$H_G^*(M) \to H_G^* \left(\mu_I^{-1}(2\pi i d \cdot \mathrm{id}) \cap \mu_{\mathbb{C}}^{-1}(0) \right)$$

surjective?

Not for $\mathcal{M}_0^{Higgs}(2,1)$ (from results of Hitchin)

Hausel/Thaddeus proved hyperkähler Kirwan surjectivity for $\mathcal{M}^{Higgs}(2,1)$ (non-fixed determinant).

Goal: (a) To use approach number 2 to study the cohomology of the moduli space of semistable Higgs bundles for rank 2 and degree 0 or degree 1.
(b) To shed some light on how this approach could work in general.

The first step is easy:

$$H^*_{\mathcal{G}}(\mu_{\mathbb{C}}^{-1}(0)) \cong H^*_{\mathcal{G}}(\mathcal{A}) \cong H^*(B\mathcal{G})$$

Then use the methods of Atiyah-Bott for the functional

$$\mathrm{YMH}(A, \phi) = \|F_A + [\phi, \phi^*]\|^2 = \|\mu_I\|^2$$

on the singular space $\mu_{\mathbb{C}}^{-1}(0)$.

Does the gradient flow converge?

What are the critical sets?

What is a sensible definition of "Morse index" on a singular space?

How to account for the singularities in the space when using Morse theory?

The critical sets of YMH are those for which the bundle splits $E = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ according to the eigenvalues of $F_A + [\phi, \phi^*]$, and the splitting is preserved by both the holomorphic structure d''_A and the endomorphism ϕ.

Theorem 1 (W)**.** *For any initial condition $(A, \phi) \in \mu_{\mathbb{C}}^{-1}(0)$, the gradient flow of $\mathsf{YMH}(A, \phi)$ converges smoothly to a critical point of $\mathsf{YMH}(A, \phi)$.*

Moreover the gradient flow preserves the ϕ-invariant Harder-Narasimhan-Seshadri filtration of (A, ϕ), and converges to a Higgs bundle isomorphic to the graded object of this filtration.

The gradient flow defines a $\mathcal{G}$-equivariant stratification of the space of Higgs bundles $\mu_{\mathbb{C}}^{-1}(0)$, with a continuous retraction of each stratum onto its associated critical set. Moreover, certain properties of a neighbourhood of each stratum are preserved by this gradient flow.

Critical points for rank 2

For rank 2, the non-minimal critical points are parametrised by the positive integers.

At the critical points:

- E splits into line bundles preserved by d_A and ϕ, $E = L_1 \oplus L_2$ with $\deg(L_1) > \deg(L_2)$ w.l.o.g.

- $\deg(L_1) = d_1$, $\deg(L_2) = d - d_1$ where d_1 is constant on each critical set.

- After dividing out by the action of $\mathcal{G}$ the critical sets are $T^*J(X) \times T^*J(X)$

(In the Atiyah & Bott case the answer is $J(X) \times J(X)$), where $J(X)$ is the Jacobian of the curve X).

Negative eigenvalues of the Hessian

Theorem 2 (W)**.** *At a critical point where* $E \cong L_1 \oplus L_2$ *the dimension of the space of negative eigenvalues of the Hessian is*

$$\dim \left(H^{0,1}(L_1^* L_2) \oplus H^{1,0}(L_1^* L_2) \right)$$

$\dim H^{0,1}(L_1^* L_2)$ corresponds exactly to the index in the Atiyah-Bott case.

$\dim H^{1,0}(L_1^* L_2)$ corresponds to the negative directions normal to the inclusion $\mathcal{A} \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)$.

- This is not well-defined on the critical set (it depends on the choice of holomorphic structure d''_A)

- For rank 2 this term is only non-zero on the first $g-1$ critical sets.

Hitchin's answer using the S^1 action on $\mathcal{M}_0^{Higgs}(2,1)$

$$P_t(\mathcal{M}_0^{Higgs}(2,1)) = P_t(\mathcal{M}_0(2,1)) + \sum_{i=1}^{g-1} t^{\lambda_i} P_t(C_i)$$

where C_i is the i^{th} critical set for the moment map $\|\phi\|^2$ associated to the S^1 action on $\mathcal{M}^{Higgs}$.

From Atiyah & Bott:

$$P_t(\mathcal{M}_0(2,1)) = P_t(B\mathcal{G}) - \frac{1}{(1-t^2)} \sum_{i=1}^{\infty} t^{\lambda_i} P_t(J(X))$$

where $\lambda_i = \dim H^{0,1}(L_1^* L_2)$ ("well-defined" directions).

Putting these results together:

$$P_t^{\mathcal{G}}(\mu_1^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))$$
$$= P_t(B\mathcal{G}) - \frac{1}{(1-t^2)} \sum_{i=1}^{\infty} t^{\lambda_i} P_t(T^* J(X))$$
$$+ \sum_{i=1}^{g-1} t^{\lambda_i} P_t(C_i)$$

The first two terms look like the answer for equivariantly perfect Morse theory with the "well-defined" directions.

How to account for "correction terms" $\sum_{i=1}^{g-1} t^{\lambda_i} P_t(C_i)$ by using an Atiyah & Bott / Kirwan-type Morse theory approach?

How to get the $+$ sign for these "correction terms"?

Can we express the correction terms by using the "extra" directions from the index calculation?

By understanding the singularities in the space $\mu_{\mathbb{C}}^{-1}(0)$ and the singularities in space of negative eigenvalues of the Hessian, we can use a Morse-theoretic approach to prove the following.

Theorem 3 (Daskalopoulos, Weitsman, W)**.** *For the case of rank* 2 *Higgs bundles with non-fixed determinant (degree zero or degree* 1*) the hyperkähler Kirwan map*

$$\kappa_H : H^*_{\mathcal{G}}\left(\mathcal{A} \times \Omega^{1,0}(\mathrm{End}(E))\right) \to H^*_{\mathcal{G}}(\mu_1^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))$$

is surjective.

For the fixed-determinant case, κ_H *is surjective up to (but not including) dimension* $4g-2$ *(rank* 2 *degree zero case) and dimension* $4g-3$ *(rank* 2 *degree one case).*

Theorem 4 (Daskalopoulos, Weitsman, W)**.** *For the case of rank* 2 *fixed determinant Higgs bundles with degree zero*

$$
\begin{aligned}
&P_t^{\mathcal{G}}(\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)) \\
&= \frac{(1+t^3)^{2g} - (1+t)^{2g} t^{2g+2}}{(1-t^2)(1-t^4)} \\
&\quad - t^{4g-4} + \frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + \frac{(1-t)^{2g} t^{4g-4}}{4(1+t^2)} \\
&\quad + \frac{(1+t)^{2g} t^{4g-4}}{2(1-t^2)} \left(\frac{2g}{t+1} + \frac{1}{t^2-1} - \frac{1}{2} + (3-2g) \right) \\
&\quad + \frac{1}{2}(2^{2g}-1) t^{4g-2} \left((1+t)^{2g-2} + (1-t)^{2g-2} - 2t^{2g-2} \right)
\end{aligned}
$$

Theorem 5 (Daskalopoulos, Weitsman, W)**.** *For the case of rank* 2 *non-fixed determinant Higgs bundles with degree zero*

$$
\begin{aligned}
&P_t^{\mathcal{G}}(\mu_1^{-1}(0)\cap\mu_{\mathbb{C}}^{-1}(0)) \\
&= \frac{(1+t)^{2g}}{(1-t^2)^2(1-t^4)}\big((1+t^3)^{2g}-(1+t)^{2g}t^{2g+2}\big) \\
&\ + \frac{(1+t)^{2g}}{1-t^2}\left(-t^{4g-4}+\frac{t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}+\frac{(1-t)^{2g}t^{4g-4}}{4(1+t^2)}\right) \\
&\ + \frac{(1+t)^{4g}t^{4g-4}}{2(1-t^2)^2}\left(\frac{2g}{t+1}+\frac{1}{t^2-1}-\frac{1}{2}+(3-2g)\right)
\end{aligned}
$$

Sketch of proof

Consider the following decomposition of an ε-neighbourhood of each stratum S_d.

Let M_d be the union of the first d strata.

Use the gradient flow to deformation retract onto the space of negative directions.

Then define:

$$M_{d\varepsilon} = S_d \cup \{\text{negative directions}\}$$

$$M_{d-1_\varepsilon} = M_{d\varepsilon} \setminus S_d$$

$$M'_{d-1_\varepsilon} = M_{d-1_\varepsilon} \setminus \{\text{"extra" directions}\}$$

Recall that for equivariantly perfect Morse theory (when $M_{d-1_\varepsilon} = M'_{d-1_\varepsilon}$), at a critical point of index λ we have the following commutative diagram as part of the LES of the pair $(M_{d\varepsilon}, M_{d-1_\varepsilon})$:

$$\begin{array}{ccccc}
H^k_{\mathcal{G}}(M_d, M_{d-1}) & \xrightarrow{\alpha^k} & H^k_{\mathcal{G}}(M_d) & \xrightarrow{\beta^k} & H^k_{\mathcal{G}}(M_{d-1}) \\
(exc.)\downarrow\cong & & \downarrow & & \downarrow \\
H^k_{\mathcal{G}}(M_{d\varepsilon}, M_{d-1_\varepsilon}) & \xrightarrow{\alpha^k_\varepsilon} & H^k_{\mathcal{G}}(M_{d\varepsilon}) & \xrightarrow{\beta^k_\varepsilon} & H^k_{\mathcal{G}}(M_{d-1_\varepsilon}) \\
(Thom)\downarrow\cong & & \downarrow & & \\
H^{(k-\lambda)}_{\mathcal{G}}(S_d) & \xrightarrow{\cup e} & H^k_{\mathcal{G}}(S_d) & &
\end{array}$$

Atiyah-Bott lemma $\Rightarrow$ $\cup e$ is injective.

Therefore α^k is injective and the top LES splits into short exact sequences.

$$H^k_{\mathcal{G}}(M_d) \cong H^k_{\mathcal{G}}(M_d, M_{d-1}) \oplus H^k_{\mathcal{G}}(M_{d-1})$$

If $M_{d-1_\varepsilon} \neq M'_{d-1_\varepsilon}$ (ie the "extra directions" exist), consider the combination of vertical and horizontal long exact sequences:

$$
\begin{array}{ccccc}
\vdots & & & & \\
\big\downarrow \delta^{k-1} & & & & \\
H^k_{\mathcal{G}}(M_d, M_{d-1}) & \xrightarrow{\alpha^k} & H^k_{\mathcal{G}}(M_d) & \xrightarrow{\beta^k} & H^k_{\mathcal{G}}(M_{d-1}) \\
(exc.)\big\downarrow\cong & & \big\downarrow & & \big\downarrow \\
H^k_{\mathcal{G}}(M_{d\varepsilon}, M_{d-1_\varepsilon}) & \xrightarrow{\alpha^k_\varepsilon} & H^k_{\mathcal{G}}(M_{d\varepsilon}) & \xrightarrow{\beta^k_\varepsilon} & H^k_{\mathcal{G}}(M_{d-1_\varepsilon}) \\
\big\downarrow \zeta^k & & \big\downarrow & & \\
H^k_{\mathcal{G}}(M_{d\varepsilon}, M'_{d-1_\varepsilon}) & \xrightarrow{\cup e} & H^k_{\mathcal{G}}(S_d) & & \\
\big\downarrow \eta^k & & & & \\
H^k_{\mathcal{G}}(M_{d-1_\varepsilon}, M'_{d-1_\varepsilon}) & & & & \\
\big\downarrow \delta^k & & & & \\
\vdots & & & &
\end{array}
$$

Lemma 6 (Daskalopoulos, Weitsman, W)**.**

$$\dim \ker \alpha^k = \dim \ker \zeta^k$$

1. By diagram chasing:

$$\begin{aligned}\dim H^k_{\mathcal{G}}(M_d) - \dim H^k_{\mathcal{G}}(M_{d-1}) \\ = \dim H^k_{\mathcal{G}}(M_{d\varepsilon}, M'_{d-1_\varepsilon}) \\ - \dim H^k_{\mathcal{G}}(M_{d-1_\varepsilon}, M'_{d-1_\varepsilon})\end{aligned}$$

Therefore if we can calculate $\dim H^k_{\mathcal{G}}(M_{d\varepsilon}, M'_{d-1_\varepsilon})$ and $\dim H^k_{\mathcal{G}}(M_{d-1_\varepsilon}, M'_{d-1_\varepsilon})$ for each value of d then we can calculate the cohomology of the minimum in terms of the critical points and the total space.

For Higgs bundles:

$$\begin{aligned}P_t^{\mathcal{G}}(\mu_1^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)) = P_t(B\mathcal{G}) - \sum_{d=1}^{\infty} P_t^{\mathcal{G}}(M_{d\varepsilon}, M'_{d-1_\varepsilon}) \\ + \sum_{d=1}^{\infty} P_t^{\mathcal{G}}(M_{d-1_\varepsilon}, M'_{d-1_\varepsilon})\end{aligned}$$

2. If ζ^k is injective for all k (ie the vertical LES splits) then α^k is injective for all k (ie the horizontal LES splits), which implies surjectivity of

$$H^k_{\mathcal{G}}(M_d) \to H^k_{\mathcal{G}}(M_{d-1})$$

for all k.

Proposition 7 (Daskalopoulos, Weitsman, W)**.** *For rank* 2 *non-fixed determinant Higgs bundles (both degree zero and degree* 1*) the vertical LES splits for every value of* d*.*

Therefore we have hyperkähler Kirwan surjectivity

$$H^*_{\mathcal{G}}(T^*\mathcal{A}) \to H^*_{\mathcal{G}}\left(\mu_1^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right)$$

in these cases.

Related cases

Semistable Bundles (Atiyah & Bott) - Symplectic reduction of the affine space of holomorphic structures on a bundle E

Semistable Higgs Bundles (Hitchin) - Hyperkähler reduction of the cotangent bundle of the affine space of holomorphic structures on a bundle E

Here we saw a relationship between the results for symplectic reduction of the space of connections and hyperkähler reduction of the cotangent bundle of the space of connections - the difference between these results involved analysing the "extra directions" from the index calculation.

Further questions

Can we follow this process for higher rank Higgs bundles? Quiver varieties?

Bradlow spaces? $U(p,q)$ Higgs bundles? (not hyperkähler but formally similar)

Can we use these ideas to compute homotopy groups of the space of strictly stable points in $\mathcal{M}_0^{Higgs}(2,0)$, which corresponds to the space of *irreducible* representations

$$\{\pi_1(X) \to SL(2,\mathbb{C})\}/SL(2,\mathbb{C})$$

Could we compute the topology of these quotients using the function

$$\|\mu_I\|^2 + \|\mu_J\|^2 + \|\mu_K\|^2$$