

THE TOPOLOGY OF THE SPACE OF STRICTLY STABLE REPRESENTATIONS OF A QUIVER

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ABSTRACT. The purpose of this paper is to describe the topology of the space of strictly stable representations of a quiver with fixed dimension vector, where stability is defined with respect to a given parameter (which may be generic or non-generic). The main result is that the homotopy groups of this space are trivial up to a certain dimension, which depends on the quiver, the choice of dimension vector, and the choice of parameter. As a corollary we also compute low-dimensional homotopy groups of the moduli space of stable representations of the quiver with fixed dimension vector, and apply the theory to the space of non-degenerate polygons in \mathbb{R}^3 .

1. INTRODUCTION

Symplectic quotients are important examples of symplectic manifolds that arise in many areas of mathematics. When the quotient is a smooth manifold there is a well-developed research program begun by Atiyah and Bott in [2], and Kirwan in [19], to compute the cohomology of these quotients via equivariant Morse theory. When the quotient is singular then in many cases of interest one would like to compute topological invariants of the smooth locus of this space, for which new techniques are needed. In the case of holomorphic bundles over a compact Riemann surface, an approach due to Daskalopoulos and Uhlenbeck is to use transversality theory to compute low-dimensional homotopy groups of the moduli space of stable holomorphic bundles. In this paper we extend the Daskalopoulos-Uhlenbeck techniques to the space of stable representations of a quiver, and compute homotopy groups of these moduli spaces.

Let $\text{Rep}(Q, \mathbf{v})$ denote the space of representations of a quiver with fixed dimension vector \mathbf{v} , and let $G_{\mathbf{v}}$ be the associated group acting on $\text{Rep}(Q, \mathbf{v})$ (see Section 2 for the precise definition). For a given stability parameter $\alpha \in \mathfrak{g}_{\mathbf{v}}^*$, denote the subset of strictly α -stable points by $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$, and the α -semistable points by $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$. In a typical situation, for almost all choices of α we have $\text{Rep}(Q, \mathbf{v})^{\alpha-st} = \text{Rep}(Q, \mathbf{v})^{\alpha-ss}$, and for these values of α the geometric invariant theory quotient $\text{Rep}(Q, \mathbf{v})//_{\alpha}G^{\mathbb{C}}$ has at worst orbifold singularities.

There are cases of interest where one would like to instead choose a value of α where $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ does not co-incide with $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$, and so the quotient is singular. In this case there are three different quotient varieties of interest: the GIT quotient $\mathcal{M}_{\alpha}(Q, \mathbf{v}) = \text{Rep}(Q, \mathbf{v})//_{\alpha}G^{\mathbb{C}}$, the resolution of singularities $\tilde{\mathcal{M}}_{\alpha}(Q, \mathbf{v})$ of the GIT quotient (as studied by Kirwan in [20]), and the smooth

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locus $\mathcal{M}_\alpha(Q, \mathbf{v})^{st} = \text{Rep}(Q, \mathbf{v})^{\alpha-st}/G^{\mathbb{C}} \subset \mathcal{M}_\alpha(Q, \mathbf{v})$. The object of study in this paper is the space $\mathcal{M}_\alpha(Q, \mathbf{v})^{st}$.

The moduli space of semistable holomorphic bundles of degree zero over a compact Riemann surface is an example of an infinite-dimensional Kähler quotient for which the Kähler quotient of the space of holomorphic structures is singular, and one would like to study the topology of the moduli space of strictly stable bundles. In the rank 2 case Kirwan applied her general theory for computing intersection Betti numbers of GIT quotients to obtain results about the low-dimensional Betti numbers (see [16] and Remark 3.4 in [17]), and later Daskalopoulos in [5] used infinite-dimensional methods in the spirit of Atiyah and Bott to obtain results about the homotopy groups of this space. Following this, Daskalopoulos and Uhlenbeck in [6] used an infinite-dimensional transversality argument to extend these results to the moduli space of stable bundles for any rank and degree, and obtained a simple formula for the homotopy groups of the moduli space of stable bundles up to a dimension bound depending on the rank of the bundle and the genus of the surface.

In this paper we extend the Daskalopoulos and Uhlenbeck approach to the moduli space of α -stable representations of a quiver with fixed dimension vector, as introduced by King in [14] (see Section 2 for more information). The difference between the approach of this paper and Kirwan's approach of [16] is twofold: (a) Kirwan's results do not apply *a priori* to the non-compact spaces $\text{Rep}(Q, \mathbf{v})$ under consideration here, one first needs to prove a gradient flow convergence result for the norm-square of the moment map on $\text{Rep}(Q, \mathbf{v})$ (which was done in [8]) as well as on the blow-ups of $\text{Rep}(Q, \mathbf{v})$ required to desingularise the quotient (the proof of which has not yet been carried out), and (b) in this paper the dimensional bound d_{min} can be computed directly by studying the different possible destabilising subrepresentations, whereas in Kirwan's method one needs to also study the desingularised spaces, the calculations for which become much more complicated as the dimension vector \mathbf{v} increases in size. Of course, it should be mentioned that Kirwan's methods do apply to a general class of quotients of compact Kähler manifolds and appear as a corollary of her results on the intersection cohomology of Kähler quotients, while the scope of this paper is limited to computing homotopy groups of Kähler quotients of $\text{Rep}(Q, \mathbf{v})$.

The main result of this paper is a computation of the low-dimensional homotopy groups of the spaces $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ and $\mathcal{M}_\alpha(Q, \mathbf{v})^{st}$, up to a dimension that depends on the underlying quiver, the choice of dimension vector, and the stability parameter α . The following is the main theorem.

Theorem 1.1. *[Theorem 3.9] Let Q be a quiver, $\text{Rep}(Q, \mathbf{v})$ be the space of representations with fixed dimension vector \mathbf{v} , let $\alpha \in \mathfrak{g}_{\mathbf{v}}^*$ be a stability parameter, and let $d_{min}(Q, \mathbf{v}, \alpha)$ be the associated minimal dimension as defined in Definition 3.1. Then $\pi_n(\text{Rep}(Q, \mathbf{v})^{\alpha-st}) = 0$ for all $n < d_{min}(Q, \mathbf{v}, \alpha) - 1$.*

Corollary 1.2. *[Corollary 3.10] Let $PG_{\mathbf{v}} = G_{\mathbf{v}}/U(1)$ be the quotient of $G_{\mathbf{v}}$ by the subgroup consisting of scalar multiples of the identity. Then $\pi_n(\mathcal{M}_\alpha(Q, \mathbf{v})^{st}) \cong \pi_{n-1}(PG_{\mathbf{v}})$ for all $n < d_{min}(Q, \mathbf{v}, \alpha) - 1$.*

As an application of the theory, some examples of interest are explicitly worked out in Section 4. The first example is a space that contains the moduli space of framed instantons on S^4 , via the ADHM construction of [3], and while the ADHM moduli space itself (which is a hyperkähler quotient, or alternately a Kähler quotient of a singular space) is beyond the scope of this paper, one can view these results as a first step towards studying the topology of this space. Again, it should be mentioned that in [18], Kirwan takes a different approach to the problem of studying the ADHM moduli space, by first desingularising the hyperkähler quotient of the space of semistable points, and then using general results (first derived in [15]) to describe the effect on low-dimensional homotopy groups of removing the strata corresponding to the singular points in the original moduli space.

The second example is the space of *non-degenerate* polygons in \mathbb{R}^3 (i.e. those polygons that do not lie in a line) modulo rigid motions of \mathbb{R}^3 . In Section 4.2 we compute the homotopy groups of the moduli space of non-degenerate polygons up to dimension 2 under certain conditions on the stability parameter.

Ultimately one would like to apply these methods to hyperkähler quotients, in particular the Nakajima quiver varieties from [27], and the moduli space of stable Higgs bundles of rank n and degree d over a compact Riemann surface from [11]. These can be viewed in two different ways: As a hyperkähler quotient $\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/G$, or as a GIT quotient of a singular space $(\mu_2^{-1}(0) \cap \mu_3^{-1}(0)) // G^{\mathbb{C}}$. In the first case there is no known stability condition, and in the second case the space $\mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ is singular, and so *a priori* the methods of this paper do not apply to this situation. It would be a very interesting problem to extend these techniques to stratified singular spaces in order to study hyperkähler quotients such as those described above.

This paper is organised as follows: Section 2 contains the background material and notational conventions used in this paper, Section 3 contains the proof of the main theorem, and the explicit examples are worked out in Section 4.

2. BACKGROUND AND NOTATION

A *quiver* Q is a directed graph, with vertices \mathcal{J} , edges \mathcal{E} and head/tail maps $h, t : \mathcal{J} \rightarrow \mathcal{E}$ that define the orientation of the edges. A *complex representation* of a quiver consists of a choice of complex vector spaces $\{V_j\}_{j \in \mathcal{J}}$ with a fixed Hermitian structure, and homomorphisms $\{A_a : V_{t(a)} \rightarrow V_{h(a)}\}_{a \in \mathcal{E}}$. The *dimension vector* of a representation is the vector $\mathbf{v} = (\dim V_j)_{j \in \mathcal{J}}$. The space of representations with fixed dimension vector is denoted

$$(2.1) \quad \text{Rep}(Q, \mathbf{v}) = \bigoplus_{a \in \mathcal{E}} \text{Hom}(V_{t(a)}, V_{h(a)}),$$

and the vector space consisting of the direct sum of all the vector spaces at each vertex is denoted

$$\text{Vect}(Q, \mathbf{v}) = \bigoplus_{j \in \mathcal{J}} V_j.$$

A *subrepresentation* of a representation $A \in \text{Rep}(Q, \mathbf{v})$ consists of vector spaces $\{V'_j\}_{j \in \mathcal{J}}$ and homomorphisms $\{A_a : V'_{t(a)} \rightarrow V'_{h(a)}\}_{a \in \mathcal{E}}$ such that $V'_j \subseteq V_j$ for all $j \in \mathcal{J}$. For each vertex $j \in \mathcal{J}$ a projection $\pi_j : V_j \rightarrow V'_j$ is defined using orthogonal projection with respect to the Hermitian structure on V_j . If the dimension vector of the subrepresentation is $\mathbf{v}' = (\dim V'_j)_{j \in \mathcal{J}}$ then we write $\mathbf{v}' \leq \mathbf{v}$, the projection $\pi : \text{Vect}(Q, \mathbf{v}) \rightarrow \text{Vect}(Q, \mathbf{v}')$ is used to denote the vector spaces in the subrepresentation, and $A' \in \text{Rep}(Q, \mathbf{v}')$ is used to denote the induced homomorphisms $A'_a = \pi \circ A_a \circ \pi \in \text{Hom}(V'_{t(a)}, V'_{h(a)})$.

Given a fixed Hermitian structure on each vector space V_j , the group $G_{\mathbf{v}} = \times_{j \in \mathcal{J}} \text{U}(V_j)$ acts on $\text{Rep}(Q, \mathbf{v})$ by

$$(2.2) \quad g \cdot \{A_a\}_{a \in \mathcal{E}} = \{g_{h(a)} A_a g_{t(a)}^{-1}\}_{a \in \mathcal{E}},$$

where $g = (g_j)_{j \in \mathcal{J}} \in \times_{j \in \mathcal{J}} \text{U}(V_j)$. This action preserves the symplectic structure (in fact the Kähler structure) on $\text{Rep}(Q, \mathbf{v})$ induced by the Hermitian structure on each V_j , and a moment map for this action is a map $\mu : \text{Rep}(Q, \mathbf{v}) \rightarrow \mathfrak{g}_{\mathbf{v}}^* = \bigoplus_{j \in \mathcal{J}} \mathfrak{u}(V_j)^*$, where the component corresponding to $\mathfrak{u}(V_j)^*$ is

$$(2.3) \quad \mu_j(A) = \sqrt{-1} \sum_{\substack{a \in \mathcal{E} \\ h(a)=j}} A_a A_a^* - \sqrt{-1} \sum_{\substack{a \in \mathcal{E} \\ t(a)=j}} A_a^* A_a.$$

For convenience, in the rest of the paper the notation

$$(2.4) \quad \mu(A) = \sqrt{-1} \sum_{a \in \mathcal{E}} [A_a, A_a^*] \in \mathfrak{g}_{\mathbf{v}}^* \subset \text{End}(\text{Vect}(Q, \mathbf{v}))$$

is used, where each $A_a \in \text{Hom}(V_{t(a)}, V_{h(a)}) \subseteq \text{End}(\text{Vect}(Q, \mathbf{v}))$, and the commutator of two elements $X, Y \in \text{End}(\text{Vect}(Q, \mathbf{v}))$ is denoted $[X, Y]$.

A central element α of $\mathfrak{g}_{\mathbf{v}}^*$ consists of a choice of a real number α_j at each vertex $j \in \mathcal{J}$, denoted

$$\alpha = (\sqrt{-1} \alpha_j \text{id}_{V_j})_{j \in \mathcal{J}}.$$

The Kähler quotient

$$(2.5) \quad \mathcal{M}_{\alpha}(Q, \mathbf{v}) = \text{Rep}(Q, \mathbf{v}) //_{\alpha} G_{\mathbf{v}} := \mu^{-1}(\alpha) / G_{\mathbf{v}}$$

is called the *representation variety*, and it depends on the quiver Q , the dimension vector \mathbf{v} , and the choice of $\alpha \in \mathfrak{g}_{\mathbf{v}}^*$.

Alternately, following the construction of Alastair King in [14], this quotient can be described via Geometric Invariant Theory as follows. Let $\alpha = (\sqrt{-1} \alpha_j \text{id}_{V_j})_{j \in \mathcal{J}}$, with each $\alpha_j \in \mathbb{Z}$, and let $G_{\mathbf{v}}^{\mathbb{C}} = \times_{j \in \mathcal{J}} \text{GL}(V_j, \mathbb{C})$ denote the complexification of G . The action of the group $G_{\mathbf{v}}^{\mathbb{C}}$ lifts to an action on the trivial line bundle $\text{Rep}(Q, \mathbf{v})$ by

$$g \cdot (\{A_a\}_{a \in \mathcal{E}}, \xi) = \left(\left\{ g_{h(a)} A_a g_{t(a)}^{-1} \right\}, \chi(g) \xi \right),$$

where the character χ is given by

$$\chi(g) = \prod_{j \in \mathcal{J}} \det(g_j)^{-\alpha_j}.$$

A representation $A \in \text{Rep}(Q, \mathbf{v})$ is α -semistable if (given some nonzero $\xi \in \mathbb{C}$) the closure of the $G_{\mathbf{v}}^{\mathbb{C}}$ -orbit of (A, ξ) in $\text{Rep}(Q, \mathbf{v}) \times \mathbb{C}$ does not intersect the zero section, i.e.

$$\overline{G_{\mathbf{v}}^{\mathbb{C}} \cdot A} \cap (\text{Rep}(Q, \mathbf{v}) \times \{0\}) = \emptyset.$$

A representation $A \in \text{Rep}(Q, \mathbf{v})$ is α -stable if (given some nonzero $\xi \in \mathbb{C}$) the $G_{\mathbf{v}}^{\mathbb{C}}$ -orbit of (A, ξ) in $\text{Rep}(Q, \mathbf{v}) \times \mathbb{C}$ is closed.

Proposition 3.1 in [14] shows that the definitions given above are equivalent to the following slope-stability conditions (analogous to those for holomorphic bundles over a Kähler manifold). Given a subrepresentation with dimension vector $\mathbf{v}' = (v_j)_{j \in \mathcal{J}}$, the rank and α -degree of the subrepresentation are given by

$$(2.6) \quad \text{rank}(Q, \mathbf{v}') = \sum_{j \in \mathcal{J}} v_j$$

$$(2.7) \quad \text{deg}_{\alpha}(Q, \mathbf{v}') = \sum_{j \in \mathcal{J}} -\alpha_j v_j$$

The α -slope is then defined to be the quotient $\text{slope}_{\alpha}(Q, \mathbf{v}') = \text{deg}_{\alpha}(Q, \mathbf{v}') / \text{rank}(Q, \mathbf{v}')$.

Lemma 2.1 (Proposition 3.1 of [14]). *A representation $A \in \text{Rep}(Q, \mathbf{v})$ is α -stable (resp. α -semistable) if every subrepresentation satisfies*

$$(2.8) \quad \text{slope}_{\alpha}(Q, \mathbf{v}') < \text{slope}_{\alpha}(Q, \mathbf{v}) \quad (\text{resp. } \text{slope}_{\alpha}(Q, \mathbf{v}') \leq \text{slope}_{\alpha}(Q, \mathbf{v})).$$

The space of α -stable (resp. α -semistable) representations is denoted $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ (resp. $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$), and the GIT quotient is defined to be

$$(2.9) \quad \mathcal{M}_{\alpha}(Q, \mathbf{v}) = \text{Rep}(Q, \mathbf{v})^{\alpha-ss} // G_{\mathbf{v}}^{\mathbb{C}},$$

where the quotient $//$ identifies orbits whose closures intersect.

Lemma 4.33 in [8] shows that the Kähler quotient defined in (2.5) can be identified with the GIT quotient defined in (2.9).

Definition 2.2. A parameter α is called *generic* if $\text{Rep}(Q, \mathbf{v})^{\alpha-st} = \text{Rep}(Q, \mathbf{v})^{\alpha-ss}$.

When α is not generic then the sets $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ and $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$ do not co-incide and the quotient may be singular. In these cases there is a smooth structure on the quotient of the subset of stable representations

$$(2.10) \quad \mathcal{M}_{\alpha}(Q, \mathbf{v})^{st} = \text{Rep}(Q, \mathbf{v})^{\alpha-st} / G_{\mathbf{v}}^{\mathbb{C}}.$$

When the parameter α is generic and the sets $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ and $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$ co-incide then the results of [8] show that topological invariants of $\mathcal{M}_{\alpha}(Q, \mathbf{v})^{st}$ can be computed using the equivariant Morse theory of Atiyah and Bott [2] and Kirwan [19]. The purpose of this paper is to understand the low-dimensional topology of $\mathcal{M}_{\alpha}(Q, \mathbf{v})^{st}$ in the cases where $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ and $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$ do not co-incide. In these cases the Morse theory of [8] only gives information about the topology of $\text{Rep}(Q, \mathbf{v})^{\alpha-ss}$, and different methods are needed to study $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ and the quotient $\mathcal{M}_{\alpha}(Q, \mathbf{v})^{st}$.

This situation is an extension to the case of quivers of the infinite-dimensional problem of stable holomorphic bundles over a compact Riemann surface studied in [5] and [6]. For rank 2 bundles Daskalopoulos used an algebraic stratification of the space of semistable bundles to obtain results about the low-dimensional topology of the space of strictly stable bundles (see Section 7 of [5] for more details). As noted on pp744-745 of [5], for higher rank bundles a stratification argument quickly becomes intractable, and in [6] a more general approach using transversality is adopted. In this paper we follow the technique of [6], and use a transversality argument to obtain the results of Theorem 3.9.

Given a representation $A \in \text{Rep}(Q, \mathbf{v})$, the *infinitesimal action* of $G_{\mathbf{v}}^{\mathbb{C}}$ at A is given by

$$(2.11) \quad \begin{aligned} \rho_A^{\mathbb{C}} : \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}} &\rightarrow T_A \text{Rep}(Q, \mathbf{v}) \\ u &\mapsto \left. \frac{d}{dt} \right|_{t=0} e^{tu} \cdot A = ([u, A_a])_{a \in \mathcal{E}}. \end{aligned}$$

Given $A \in \text{Rep}(Q, \mathbf{v})^{\alpha-st}$, the infinitesimal deformations of the moduli space $\mathcal{M}_{\alpha}(Q, \mathbf{v})^{st}$ at the equivalence class containing A can be viewed in terms of the cohomology of a complex

$$(2.12) \quad \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}} \xrightarrow{\rho_A} T_A \text{Rep}(Q, \mathbf{v})$$

The tangent space to the moduli space is isomorphic to $T_A \text{Rep}(Q, \mathbf{v}) / \text{im } \rho_A = \text{coker } \rho_A$. Since the action of $G_{\mathbf{v}}^{\mathbb{C}}$ is free at an α -stable point, then $\ker \rho_A = \{0\}$, and so

$$\dim_{\mathbb{C}} \text{coker } \rho_A = \dim_{\mathbb{C}} \text{coker } \rho_A - \dim_{\mathbb{C}} \ker \rho_A = \dim_{\mathbb{C}} T_A \text{Rep}(Q, \mathbf{v}) - \dim_{\mathbb{C}} \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}}.$$

When $A \in \text{Rep}(Q, \mathbf{v})$ is α -unstable, then one can still define the complex (2.12), however $\ker \rho_A$ may now be non-trivial. A particular case of interest is when A preserves the vector spaces $\text{Vect}(Q, \mathbf{v}')$ of a given subrepresentation π . In this case one can view π as an endomorphism $\pi \in \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}} = \bigoplus_{j \in \mathcal{J}} \text{End}(V_j)$, and $\rho_A(\pi) = ([A_a, \pi])_{a \in \mathcal{E}}$. If $\pi : \text{Vect}(Q, \mathbf{v}) \rightarrow \text{Vect}(Q, \mathbf{v}')$ is a subrepresentation of A (i.e. $A_a(\pi(V_{t(a)})) \subseteq \pi(V_{h(a)})$ for each $a \in \mathcal{E}$), then $\pi \circ A_a \circ \pi = A_a \circ \pi = A_a \circ \pi \circ \pi$, or alternately $[A_a, \pi] \circ \pi = 0$ for each $a \in \mathcal{E}$. Conversely, if $[A_a, \pi] \circ \pi = 0$ for each $a \in \mathcal{E}$, then A_a preserves the vector spaces in the image of the projection π . Therefore we have proved the following lemma.

Lemma 2.3. $\pi : \text{Vect}(Q, \mathbf{v}) \rightarrow \text{Vect}(Q, \mathbf{v}')$ is a subrepresentation of $A \in \text{Rep}(Q, \mathbf{v})$ iff $\rho_A(\pi) \circ \pi = 0$.

Given a subrepresentation $\pi : \text{Vect}(Q, \mathbf{v}) \rightarrow \text{Vect}(Q, \mathbf{v}')$ and the orthogonal complement $\pi^{\perp} := (\text{id} - \pi) : \text{Vect}(Q, \mathbf{v}) \rightarrow \text{Vect}(Q, \mathbf{v})^{\perp}$ we define the following subspaces of $\mathfrak{g}_{\mathbf{v}}^{\mathbb{C}}$ and $T_A \text{Rep}(Q, \mathbf{v})$.

$$\begin{aligned} \text{Hom}^0(\pi, \pi^{\perp}) &:= \bigoplus_{j \in \mathcal{J}} \text{Hom}(\pi(V_j), \pi^{\perp}(V_j)) \subseteq \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}} \\ \text{Hom}^1(\pi, \pi^{\perp}) &:= \bigoplus_{a \in \mathcal{E}} \text{Hom}(\pi(V_{t(a)}), \pi^{\perp}(V_{h(a)})) \subseteq \text{Rep}(Q, \mathbf{v}) \end{aligned}$$

Moreover, if A preserves both π and π^\perp , then we say that the representation *splits*, and there is an induced map

$$(2.13) \quad \mathrm{Hom}^0(\pi, \pi^\perp) \xrightarrow{\rho_A} \mathrm{Hom}^1(\pi, \pi^\perp),$$

where we identify $\mathrm{Rep}(Q, \mathbf{v})$ with the tangent space $T_A \mathrm{Rep}(Q, \mathbf{v})$. This viewpoint will become useful in the next section, where we use this to define a simple condition for a representation to be unstable.

3. TRANSVERSALITY RESULTS

The purpose of this section is to prove Theorem 3.9, which shows that the homotopy groups of the space $\mathrm{Rep}(Q, \mathbf{v})^{\alpha-st}$ are trivial up to a dimensional bound that depends on the quiver Q , the dimension vector \mathbf{v} and the parameter α . The key idea behind this is to show that for small values of n there exists a based homotopy equivalence F between a given map $s : S^n \rightarrow \mathrm{Rep}(Q, \mathbf{v})^{\alpha-st}$ and the constant map, such that the image of F is contained in $\mathrm{Rep}(Q, \mathbf{v})^{\alpha-st}$. Equivalently, we show that the image of F does not contain any representations that preserve a sub-representation violating the slope-stability condition of Lemma 2.1.

Given a dimension vector $\mathbf{v}' \leq \mathbf{v}$, the set of all possible subrepresentations $\pi : \mathrm{Vect}(Q, \mathbf{v}) \rightarrow \mathrm{Vect}(Q, \mathbf{v}')$ is given by

$$(3.1) \quad \mathrm{Gr}(\mathbf{v}', \mathbf{v}) := \prod_{j \in \mathcal{J}} \mathrm{Gr}(v'_j, v_j).$$

For $\pi \in \mathrm{Gr}(\mathbf{v}', \mathbf{v})$, the value of $\dim_{\mathbb{R}} \mathrm{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \mathrm{Hom}^0(\pi, \pi^\perp)$ depends only on the dimension vector \mathbf{v}' .

Definition 3.1. The *minimal dimension* of the triple (Q, \mathbf{v}, α) is

$$(3.2) \quad d_{\min}(Q, \mathbf{v}, \alpha) = \min_{\substack{\mathbf{v}' < \mathbf{v} \\ \mathrm{slope}_\alpha(Q, \mathbf{v}') \geq \mathrm{slope}_\alpha(Q, \mathbf{v})}} \dim_{\mathbb{R}} \mathrm{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \mathrm{Hom}^0(\pi, \pi^\perp).$$

Let $T \rightarrow \mathrm{Gr}(\mathbf{v}', \mathbf{v})$ denote the tautological bundle with fibre over $\pi \in \mathrm{Gr}(\mathbf{v}', \mathbf{v})$ given by $\mathrm{Vect}(Q, \mathbf{v}')$, a sub-bundle of the trivial $\mathrm{Vect}(Q, \mathbf{v})$ bundle over $\mathrm{Gr}(\mathbf{v}', \mathbf{v})$. Similarly, let $W^0 \rightarrow \mathrm{Gr}(\mathbf{v}', \mathbf{v})$ be the tautological sub-bundle of the trivial bundle $\mathrm{Gr}(\mathbf{v}', \mathbf{v}) \times \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}}$, where the fibres of W^0 are the subspaces $\mathrm{Hom}^0(\pi, \pi^\perp) \subseteq \mathfrak{g}_{\mathbf{v}}^{\mathbb{C}}$, and let $W^1 \rightarrow \mathrm{Gr}(\mathbf{v}', \mathbf{v})$ be the tautological sub-bundle of the trivial bundle $\mathrm{Gr}(\mathbf{v}', \mathbf{v}) \times \mathrm{Rep}(Q, \mathbf{v})$, where the fibres of W^1 are the subspaces $\mathrm{Hom}^1(\pi, \pi^\perp) \subseteq \mathrm{Rep}(Q, \mathbf{v})$.

Given a compact manifold X , and basepoints $x_0 \in X$ and $A_0 \in \mathrm{Rep}(Q, \mathbf{v})$, let $C^q(X, \mathrm{Rep}(Q, \mathbf{v}))$ be the set of C^q maps from X to $\mathrm{Rep}(Q, \mathbf{v})$, this has the structure of a C^q manifold. Let $\mathcal{C}(X) \subset C^q(X, \mathrm{Rep}(Q, \mathbf{v}))$ be the subset of basepoint-preserving C^q maps from X to $\mathrm{Rep}(Q, \mathbf{v})$.

Lemma 3.2. $\mathcal{C}(X)$ is a submanifold of $C^q(X, \mathrm{Rep}(Q, \mathbf{v}))$, and the tangent space at a point $s \in \mathcal{C}(X)$ is

$$(3.3) \quad T_s \mathcal{C}(X) = \{\delta s : X \rightarrow \mathrm{Rep}(Q, \mathbf{v}) : \delta s \in C^q, \delta s(x_0) = 0\}.$$

Proof. Define $L : C^q(X, \text{Rep}(Q, \mathbf{v})) \rightarrow \text{Rep}(Q, \mathbf{v})$ by $L(s) = s(x_0)$. Then $\mathcal{C}(X) = L^{-1}(A_0)$, and so by the implicit function theorem, the proof reduces to showing that A_0 is a regular value of L . The derivative of L at $s \in \mathcal{C}(X)$ is $\delta L_s(\delta s) = \delta s(x_0)$, and the Whitney extension theorem (cf. Appendix A in [1]) shows that given a prescribed first order perturbation δs at x_0 there exists $s \in C^q(X, \text{Rep}(Q, \mathbf{v}))$ with $s(x_0) = A_0$ and the first order perturbation of s at x_0 equal to δs . Therefore δL is surjective and A_0 is a regular value of L . \square

The space of basepoint-preserving homotopy equivalences between maps in $\mathcal{C}(X)$ is defined as follows. Let $\mathcal{S}(X) \subset C^q(X \times [0, 1])$ be the subset of maps $F : X \times [0, 1] \rightarrow \text{Rep}(Q, \mathbf{v})$ such that $F(x_0, t) = A_0$ for all $t \in [0, 1]$.

Lemma 3.3. *The space $\mathcal{S}(X)$ has the structure of a C^q manifold, and the tangent space at $F \in \mathcal{S}(X)$ is*

$$(3.4) \quad T_F \mathcal{S}(X) = \{\delta F : X \times [0, 1] \rightarrow \text{Rep}(Q, \mathbf{v}) : \delta F \in C^q, \delta F(x_0, t) = 0 \forall t \in [0, 1]\}.$$

The proof of this statement is similar to that of the previous lemma (which uses the implicit function theorem and Whitney extension theorem), and so it is omitted.

Given $s \in \mathcal{C}(X)$, the space of basepoint-preserving homotopy equivalences between s and the constant map is

$$(3.5) \quad \mathcal{S}(s) = \{F \in \mathcal{S}(X) : F(x, 0) = s(x), F(x, 1) = A_0, F(x_0, t) = A_0\}.$$

Since $\text{Rep}(Q, \mathbf{v})$ is a vector space then $\mathcal{S}(s)$ is clearly non-empty.

Lemma 3.4. *$\mathcal{S}(s)$ is a C^q submanifold of $\mathcal{S}(X)$, and the tangent space of $\mathcal{S}(s)$ at F is*

$$(3.6) \quad T_F \mathcal{S}(s) = \{\delta F : X \times [0, 1] \rightarrow \text{Rep}(Q, \mathbf{v}) : \delta F \in C^q, \delta F(x, 0) = \delta F(x, 1) = \delta F(x_0, t) = 0, \forall t \in [0, 1], x \in X\}.$$

The proof of this lemma is similar to that of the previous lemma (see also the proof of Lemma 2.1 in [6]), and so it is omitted.

Now fix $X = S^n$ and $s : X \rightarrow \text{Rep}(Q, \mathbf{v})^{\alpha-st}$. For notation let $\mathcal{C} = \mathcal{C}(X)$, $\mathcal{S} = \mathcal{S}(s)$ and $Y = S^n \times [0, 1]$. Given a dimension vector $\mathbf{v}' \leq \mathbf{v}$, define the map

$$\begin{aligned} \mathcal{D} : \text{Gr}(\mathbf{v}', \mathbf{v}) \times \mathcal{S} \times Y &\rightarrow W^1 \subseteq \text{Gr}(\mathbf{v}', \mathbf{v}) \times \text{Rep}(Q, \mathbf{v}) \\ \mathcal{D}(\pi, F, y) &= \left(\pi, \pi^\perp \circ \rho_{F(y)}(\pi) \circ \pi \right) \end{aligned}$$

Since the component of $\rho_{F(y)}(\pi)$ along each edge $a \in \mathcal{E}$ is given by $[F(y)_a, \pi]$, then for simplicity of notation we write $\rho_{F(y)}(\pi) = [F(y), \pi]$, where the meaning is that the component of $[F(y), \pi]$ corresponding to the edge $a \in \mathcal{E}$ is $[F(y)_a, \pi]$. For a given map $F \in \mathcal{S}(s)$, define $\mathcal{D}^F(\pi, y) = \mathcal{D}(\pi, F, y)$.

Remark 3.5. (i) Since $\pi^\perp \circ \pi = 0$ and $\pi \circ \pi = \pi$, then we can write

$$\mathcal{D}(\pi, F, y) = \left(\pi, \pi^\perp \circ [F(y), \pi] \circ \pi \right) = \left(\pi, \pi^\perp \circ F(y) \circ \pi \right).$$

- (ii) Let \mathcal{O}_{W^1} denote the zero section of W^1 . Then $F(y) \in \text{Rep}(Q, \mathbf{v})$ preserves the subspaces defined by π iff $\pi^\perp \circ F(y) \circ \pi = 0$, which by the previous statement is equivalent to saying that $F(y)$ preserves π iff $(\pi, F, y) \in \mathcal{D}^{-1}(\mathcal{O}_{W^1})$.
- (iii) The same computation as above shows that

$$F(y) \text{ preserves } \pi \iff \pi^\perp \circ [F(y), \pi] = 0 \iff [F(y), \pi] \circ \pi = 0.$$

Proposition 3.6. (i) $\mathcal{D} : \text{Gr}(\mathbf{v}', \mathbf{v}) \times \mathcal{S} \times Y \rightarrow W^1$ is a C^q map of fibre bundles over $\text{Gr}(\mathbf{v}', \mathbf{v})$.

(ii) \mathcal{D} is transverse to the zero section \mathcal{O}_{W^1} of W^1 .

(iii) Let W_π^1 be the fibre of W^1 over $\pi \in \text{Gr}(\mathbf{v}', \mathbf{v})$. Then

$$(3.7) \quad \text{codim}_{W^1} \mathcal{O}_{W^1} = \dim_{\mathbb{R}} W_\pi^1 = \dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp)$$

$$(3.8) \quad \dim_{\mathbb{R}} \text{Gr}(\mathbf{v}', \mathbf{v}) \times Y = \dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) + n + 1$$

Proof. It is clear that \mathcal{D} maps fibres of $\text{Gr}(\mathbf{v}', \mathbf{v}) \times \mathcal{S} \times Y$ to fibres of W^1 , and so (i) follows from the fact that F is C^q .

To see (ii), a calculation shows that the derivative of \mathcal{D} with respect to F at a point $(\pi, F, y) \in \mathcal{D}^{-1}(\mathcal{O}_{W^1})$ is given by

$$(3.9) \quad d_2 \mathcal{D}_{(\pi, F, y)}(\delta F) = \pi^\perp \circ \delta F \circ \pi.$$

Since $s : X \rightarrow \text{Rep}(Q, \mathbf{v})^{\alpha-st}$, $A_0 \in \text{Rep}(Q, \mathbf{v})^{\alpha-st}$, and $F \in \mathcal{S}(s)$, then $(\pi, F, y) \in \mathcal{D}^{-1}(\mathcal{O}_{W^1})$ implies that $y \notin X \times \{0\}$, $y \notin X \times \{1\}$ and $y \notin \{x_0\} \times [0, 1]$. Therefore, from (3.6), $\delta F(y)$ can take any value in $\text{Rep}(Q, \mathbf{v})$, and so the above equation shows that $d_2 \mathcal{D}_{(\pi, F, y)}$ is onto $\text{Hom}^1(\pi, \pi^\perp)$, the fibre of W^1 at π .

In the proof of (iii), the first statement follows by observing that $W_\pi^1 \cong \text{Hom}^1(\pi, \pi^\perp)$. The second statement follows from the definition of $\text{Gr}(\mathbf{v}', \mathbf{v})$ in (3.1). \square

In the following Corollary, recall that a *residual subset* of a topological space is the countable intersection of open dense subsets (cf. [1, p41]).

Corollary 3.7. *If $q \geq 1$ and $\dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) + n + 1 < \dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp)$, then for F in a residual subset of \mathcal{S} , the image $\mathcal{D}^F(\text{Gr}(\mathbf{v}', \mathbf{v}) \times \mathcal{S}^n \times [0, 1])$ does not intersect \mathcal{O}_{W^1} .*

Proof. For $(\pi, F, y) \in \mathcal{D}^{-1}(\mathcal{O}_{W^1})$, let $p : T_{\mathcal{D}(\pi, F, y)} W^1 \rightarrow T_0 W_\pi^1$ be the natural projection map. Since the manifolds $\text{Gr}(\mathbf{v}', \mathbf{v})$ and Y are finite-dimensional, as are the fibres W_π^1 of the bundle W^1 , then $p \circ (\delta \mathcal{D}^F)_{(\pi, y)} : T_\pi \text{Gr}(\mathbf{v}', \mathbf{v}) \times T_y Y \rightarrow T_0 W_\pi^1$ is Fredholm of index

$$\dim_{\mathbb{R}} \text{Gr}(\mathbf{v}', \mathbf{v}) + \dim_{\mathbb{R}} Y - \dim_{\mathbb{R}} W_\pi^1 = \dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) + n + 1 - \dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp).$$

Therefore a combination of Proposition 3.6 together with the Daskalopoulos-Uhlenbeck version of the transversal density theorem (Theorems A2 and A4 in [6]) shows that the set

$$\mathcal{S}_{\mathcal{O}_{W^1}} = \{F \in \mathcal{S} : \mathcal{D}^F \text{ is transversal to } \mathcal{O}_{W^1}\}$$

is residual in \mathcal{S} .

Since the dimension of the fibres of W^1 (or alternately, the codimension of \mathcal{O}_{W^1} in W^1) is given by $\dim \operatorname{Hom}^1(\pi, \pi^\perp)$, and by assumption we have $\dim \operatorname{Gr}(\mathbf{v}', \mathbf{v}) + n + 1 < \dim \operatorname{Hom}^1(\pi, \pi^\perp)$, then the map $\mathcal{D}^F : \operatorname{Gr}(\mathbf{v}', \mathbf{v}) \times S^n \times [0, 1] \rightarrow W$ is transverse to the zero section \mathcal{O}_{W^1} iff the image $\mathcal{D}^F(\operatorname{Gr}(\mathbf{v}', \mathbf{v}) \times S^n \times [0, 1])$ does not intersect \mathcal{O}_{W^1} . \square

Remark 3.8. This corollary shows that there is a residual set of C^1 basepoint-preserving homotopy equivalences F between $s : S^n \rightarrow \operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$ and the constant map, such that no representation in the image of F preserves a sub-representation in $\operatorname{Gr}(\mathbf{v}', \mathbf{v})$.

Theorem 3.9. *If $n + 1 < d_{\min}(Q, \mathbf{v}, \alpha)$ then $\pi_n(\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}) = 0$.*

Proof. Fix a continuous map $s : S^n \rightarrow \operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$. We aim to show that there is a basepoint-preserving homotopy equivalence in $\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$ between s and the constant map. Firstly note that since the set of stable points $\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$ is open in the vector space $\operatorname{Rep}(Q, \mathbf{v})$, then we can perturb s (while remaining in $\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$) so that $s \in C^1$ (cf. [25, Theorem 10.21]).

Corollary 3.7 then shows that for each $\mathbf{v}' < \mathbf{v}$ there is a residual set $\mathcal{S}(s)_{\mathbf{v}'}^0$ of homotopy equivalences F from s to the constant map such that no representation in the image of F preserves a subrepresentation in $\operatorname{Gr}(\mathbf{v}', \mathbf{v})$.

There are a finite number of choices of $\mathbf{v}' < \mathbf{v}$ such that $\operatorname{slope}_\alpha(Q, \mathbf{v}') \geq \operatorname{slope}_\alpha(Q, \mathbf{v})$, and so the set

$$\mathcal{S}(s)_\alpha^0 = \bigcap_{\substack{\mathbf{v}' < \mathbf{v} \\ \operatorname{slope}_\alpha(Q, \mathbf{v}') \geq \operatorname{slope}_\alpha(Q, \mathbf{v})}} \mathcal{S}(s)_{\mathbf{v}'}^0$$

is residual in $\mathcal{S}(s)$. In other words, there is a residual set $\mathcal{S}(s)_\alpha^0$ of C^1 homotopy equivalences between s and the constant map, such that for every $F \in \mathcal{S}(s)_\alpha^0$ the image $F(S^n \times [0, 1])$ is contained in $\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$.

Since residual sets are non-empty, then there is at least one basepoint-preserving homotopy equivalence in $\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$ between s and the constant map, and so $\pi_n(\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}) = 0$. \square

Corollary 3.10. *If $n + 1 < d_{\min}(Q, \mathbf{v}, \alpha)$ then $\pi_n(\mathcal{M}_\alpha(Q, \mathbf{v})^{st}) \cong \pi_{n-1}(PG_{\mathbf{v}})$.*

Proof. The group $PG_{\mathbf{v}}$ acts freely on $\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}$, and the quotient is

$$\mathcal{M}_\alpha(Q, \mathbf{v})^{st} = \operatorname{Rep}(Q, \mathbf{v})^{\alpha-st} / PG_{\mathbf{v}}.$$

The fibre bundle

$$PG_{\mathbf{v}} \rightarrow \operatorname{Rep}(Q, \mathbf{v})^{\alpha-st} \rightarrow \mathcal{M}_\alpha(Q, \mathbf{v})^{st}$$

induces a long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_n(\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}) \rightarrow \pi_n(\mathcal{M}_\alpha(Q, \mathbf{v})^{st}) \rightarrow \pi_{n-1}(PG_{\mathbf{v}}) \rightarrow \pi_{n-1}(\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}) \rightarrow \cdots,$$

therefore an application of Theorem 3.9 shows that $\pi_n(\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}) = \pi_{n-1}(\operatorname{Rep}(Q, \mathbf{v})^{\alpha-st}) = 0$ for $n < d_{\min}(Q, \mathbf{v}, \alpha) - 1$, and so the long exact sequence above gives the isomorphism

$$\pi_n(\mathcal{M}_\alpha(Q, \mathbf{v})^{st}) \cong \pi_{n-1}(PG_{\mathbf{v}}),$$

which completes the proof. \square

4. EXAMPLES

This section contains two applications of Theorem 3.9 and Corollary 3.10. The first is to study the low-dimensional topology of representation varieties that contain Nakajima quiver varieties, an explicit calculation is performed for the case of the ADHM quiver with the results in Theorem 4.1. The second application of the main theorem is to the moduli space of polygons in \mathbb{R}^3 , which parametrises polygons in \mathbb{R}^3 with a fixed number of sides (determined by the quiver), and fixed sidelengths (determined by the choice of parameter), up to equivalence by rigid motions in \mathbb{R}^3 . A non-generic choice of parameter means that certain polygons may lie in a line, such degenerate polygons correspond to singularities in the moduli space $\mathcal{M}_\alpha(Q, \mathbf{v})$ since there are rotations of \mathbb{R}^3 that leave such polygons invariant. In Theorem 4.4 we describe the low-dimensional topology of the space of *non-degenerate polygons* (those that are not contained in a line), in the case where the parameter α is non-generic.

4.1. The “Doubled” ADHM quiver. It is well-known that the moduli space of framed instantons on S^4 has a description in terms of a finite-dimensional hyperkähler quotient construction, known as the *ADHM construction* (see [3], [7] for more details). Later work of Kronheimer and Nakajima extended this construction to moduli spaces of framed instantons over the one-point compactification of an ALE hyperkähler four-manifold (see [23], [24]).

This finite-dimensional hyperkähler quotient construction is a special case of what is now known as a *Nakajima quiver variety*, where a new quiver \tilde{Q} is constructed by adding a “dual” edge to each edge on the original quiver Q , the new space of representations $\text{Rep}(\tilde{Q}, \mathbf{v})$ is now the cotangent bundle of the original space of representations, and the moduli space $\mathcal{M}_\alpha^{hk}(\tilde{Q}, \mathbf{v})$ is defined to be the hyperkähler quotient of $\text{Rep}(\tilde{Q}, \mathbf{v}) = T^*\text{Rep}(Q, \mathbf{v})$ (see [27] for a complete description). For the quiver \tilde{Q} pictured in (4.1), the moduli space of α -stable equivalence classes in the hyperkähler quotient when $\alpha = 0$ corresponds to the moduli space of framed instantons on S^4 , the Uhlenbeck compactification of this space corresponds to the moduli space of α -semistable equivalence classes, and the Gieseker compactification corresponds to the hyperkähler quotient at a generic parameter (see for example Section 5.2 of [26] for more details).

There is an inclusion of each of these moduli spaces $\mathcal{M}_{(\alpha, \beta)}^{hk}(Q, \mathbf{v}) \hookrightarrow \mathcal{M}_\alpha(\tilde{Q}, \mathbf{v})$ into the representation variety of the quiver \tilde{Q} , formed by taking the Kähler quotient instead of the hyperkähler quotient. The purpose of this section is to use Theorem 3.9 to study the topology of the space $\mathcal{M}_0(\tilde{Q}, \mathbf{v})^{st}$ in the case of the quiver corresponding to the ADHM construction of framed instantons on S^4 . The same methods will also apply to other representation varieties that contain Nakajima quiver varieties (for example moduli spaces of instantons over other ALE hyperkähler four-manifolds), one only needs to compute d_{min} for these examples to prove an analogous result.

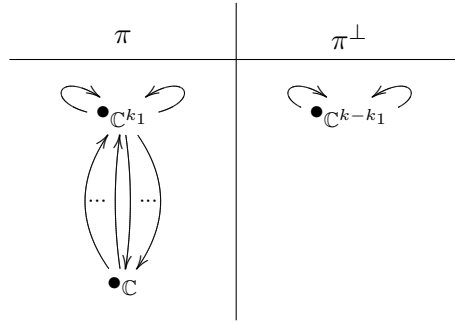
Fix a vector bundle E over S^4 with rank n and instanton number k . Then the quiver \tilde{Q} under consideration is given as follows.

(4.1) 

(note that this differs slightly from Donaldson’s description in [7], here we use the construction of Crawley-Boevey in [4] which identifies the representation variety of the quiver above with the solutions to the equations from [7]).

There are n arrows from the vertex \mathbb{C}^k to the vertex \mathbb{C} , another n arrows from the vertex \mathbb{C} to the vertex \mathbb{C}^k , and two arrows from \mathbb{C}^k to itself. In this case $G_{\mathbf{v}} = \mathrm{U}(k) \times \mathrm{U}(1)$, and $PG_{\mathbf{v}} \cong \mathrm{U}(k)$.

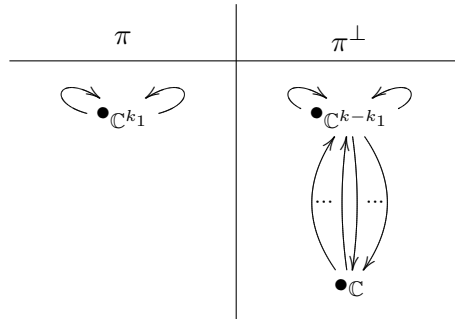
A subrepresentation π can have one of the following forms (the diagram below also includes the quiver for π^\perp). Firstly we have the case where the vertex \mathbb{C} is contained in the subrepresentation π . Since π cannot have dimension vector \mathbf{v} , then $k_1 < k$ in the diagram below.



The elements of $\mathrm{Hom}^1(\pi, \pi^\perp)$ correspond to maps from the vertex \mathbb{C}^{k_1} to \mathbb{C}^{k-k_1} (there are two edges with this property) and maps from the vertex \mathbb{C} to \mathbb{C}^{k-k_1} (there are n edges with this property). Therefore we see that $\dim_{\mathbb{R}} \mathrm{Hom}^1(\pi, \pi^\perp) = 4k_1(k - k_1) + 2n(k - k_1)$. A similar computation shows that $\dim_{\mathbb{R}} \mathrm{Hom}^0(\pi, \pi^\perp) = 2k_1(k - k_1)$. Therefore in this case

(4.2) $\dim_{\mathbb{R}} \mathrm{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \mathrm{Hom}^0(\pi, \pi^\perp) = 2(k_1 + n)(k - k_1)$.

The second case is when the vertex \mathbb{C} is contained in π^\perp . Since π cannot have dimension vector $\mathbf{0}$, then $k_1 > 0$ in the diagram below.



A calculation shows that $\dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) = 4k_1(k - k_1) + 2nk_1$ and $\dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) = 2k_1(k - k_1)$. Therefore in this case

$$(4.3) \quad \dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) = 2k_1(k - k_1 + n).$$

In both cases we see that $\dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) \geq 2(k - 1 + n)$, and so $d_{\min} \geq 2(k - 1 + n)$. Therefore Theorem 3.9 shows that

Theorem 4.1. *Let \tilde{Q} be the quiver in (4.1), with $\dim_{\mathbb{C}} V_1 = k$ and n arrows from V_2 to V_1 . Then*

$$\pi_j \left(\text{Rep}(\tilde{Q}, \mathbf{v}) \right) = 0$$

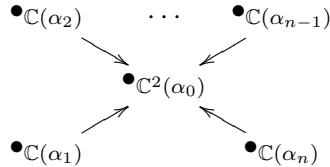
for all $j < 2(k - 1 + n) - 1$, and

$$\pi_j \left(\mathcal{M}_0(\tilde{Q}, \mathbf{v})^{st} \right) = \pi_j \left(\text{U}(k) \right)$$

for all $j < 2(k - 1 + n) - 1$.

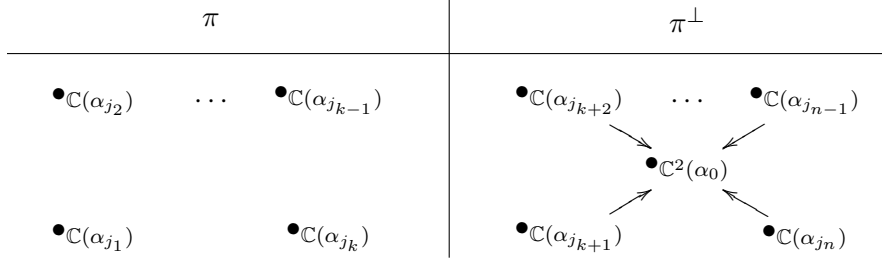
4.2. The moduli space of non-degenerate polygons in \mathbb{R}^3 . The topology and geometry of the moduli space of polygons in \mathbb{R}^3 with fixed sidelengths has been well-studied, see for example [21], [13], [10], [9]. When the sidelengths are chosen so that some polygons may lie in a line (we call these *degenerate* polygons) then the moduli space has singularities, corresponding to the fact that a degenerate polygon is invariant under a circle subgroup of the rotation group. The purpose of this section is to prove new results about the low-dimensional topology of the moduli space of *non-degenerate* polygons.

An n -sided polygon in \mathbb{R}^3 with basepoint at the origin corresponds to a choice of n vectors e_1, \dots, e_n such that $e_1 + \dots + e_n = 0$, where the vertices of the polygon are then given by $v_k = e_1 + \dots + e_k$. The *moduli space of polygons in \mathbb{R}^3 with fixed sidelengths* is the space of all polygons in \mathbb{R}^3 with basepoint at the origin and given sidelengths s_1, \dots, s_n , modulo rotations by $\text{SO}(3)$. It is well-known (see for example [22]) that this moduli space corresponds to the representation variety of a quiver, given as follows. Let Q be the “star-shaped” quiver below, with one central vertex and n outer vertices (one for each side of the polygon). Fix a dimension vector \mathbf{v} , with dimension 2 at the central vertex, and dimension 1 at each of the outer vertices. In this case $G_{\mathbf{v}} = \text{U}(2) \times \text{U}(1)^n$, and $PG_{\mathbf{v}} \cong \text{U}(2) \times \text{U}(1)^{n-1}$.



With the conventions of Section 2, the parameter α_j at the j^{th} vertex is chosen so that $\alpha_j = -s_j^2$ (where s_j is the sidelength of the j^{th} edge of the polygon in \mathbb{R}^3), and the parameter α_0 is then given by $\alpha_0 = -\frac{1}{2} \sum_{j=1}^n \alpha_j$, i.e. the associated central element $\alpha \in \mathfrak{g}_{\mathbf{v}}^*$ is trace-free.

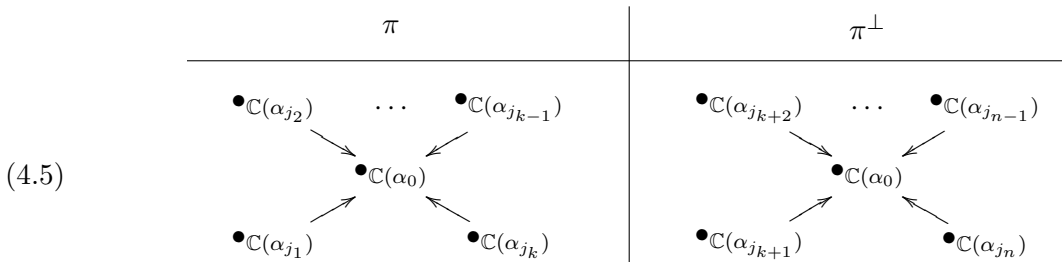
Since $\alpha_j < 0$ for each $j = 1, \dots, n$ then by the definition of slope stability in (2.8) we see that a representation which is not strictly α -stable must correspond to one of the following two cases. The first case consists of those subrepresentations π that do not contain any subspaces of the central vertex (again, the diagram below includes the subrepresentation π^\perp). The outer vertices in the following diagram have been re-labelled, so that α_{j_i} for $1 \leq i \leq k$ are in the subrepresentation π , and α_{j_i} for $k+1 \leq i \leq n$ are in the subrepresentation π^\perp .



Note that $k \geq 1$ so that the subrepresentation π is non-trivial. The elements of $\text{Hom}^1(\pi, \pi^\perp)$ correspond to maps from the vertices in the quiver on the left (corresponding to the subrepresentation π) to the central vertex in the quiver on the right (corresponding to π^\perp). A calculation shows that $\dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) = 4k$, and that $\dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) = 0$. Therefore, in this case

$$(4.4) \quad \dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) = 4k \geq 4.$$

The other possible type of subrepresentation occurs when the subrepresentation π contains a complex dimension one subspace of the central vertex, described in the diagram below.



Unlike the previous case (where any value of k between 1 and n could occur), the range of possible values of k is limited by the choice of α_j at each vertex (i.e. the sidelengths of the polygon). To see this, let \mathbf{v}' be the dimension vector associated to the subrepresentation π , and note that for the condition $\text{slope}_\alpha(Q, \mathbf{v}') \geq \text{slope}_\alpha(Q, \mathbf{v})$ to hold we must have

$$\deg(Q, \mathbf{v}') = -\alpha_0 - \sum_{i=1}^k \alpha_{j_i} \geq 0,$$

which places a restriction on the possible configurations that can occur in this case (recall that $\alpha_0 > 0$ and that $\alpha_j < 0$ for $j = 1, \dots, k$).

A calculation shows that $\dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) = 2k$, and that $\dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) = 2$. Therefore, in this case

$$(4.6) \quad \dim_{\mathbb{R}} \text{Hom}^1(\pi, \pi^\perp) - \dim_{\mathbb{R}} \text{Hom}^0(\pi, \pi^\perp) = 2k - 2.$$

In analogy with [9], define a subset $J \subset \{1, \dots, n\}$ to be *strictly short* if

$$(4.7) \quad -\sum_{j \in J} \alpha_j < -\sum_{j \notin J} \alpha_j.$$

Equivalently, J is strictly short if $-\alpha_0 - \sum_{j \in J} \alpha_j < 0$.

Definition 4.2. The parameter α is ℓ -*strictly short* if every subset $J \subset \{1, \dots, n\}$ with ℓ elements is strictly short.

Remark 4.3. If α is ℓ -strictly short, then there are no subrepresentations of the type in diagram (4.5) with $k \leq \ell$.

The previous calculations show that d_{min} takes the following values when α is ℓ -strictly short.

| | |
|-----------|--------------|
| d_{min} | ℓ |
| 0 | 0 |
| 2 | 1 |
| 4 | 2 or greater |

Together with Theorem 3.9 and Corollary 3.10, the above calculations lead to the following theorem.

Theorem 4.4. (i) *If α is 2-strictly short then*

- $\pi_j(\text{Rep}(Q, \mathbf{v})^{\alpha-st}) = 0$ for $j \leq 2$.
- $\pi_j(\mathcal{M}_\alpha(Q, \mathbf{v})^{st}) = \pi_j(U(2) \times U(1)^{n-1})$ for $j \leq 2$.

(ii) *If α is 1-strictly short then $\mathcal{M}_\alpha(Q, \mathbf{v})^{st}$ and $\text{Rep}(Q, \mathbf{v})^{\alpha-st}$ are both connected.*

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