Equivalent Conditions of Riemann Integrable Functions

In this note, I revised the proof of Lemma 6.2.1 and Theorem 6.2.1. I hope the revised proof will be easier to understand.

**Lemma 6.2.1** If the maximal interval length of $P'$ is less than the minimum length of subintervals of $P$ (that is, $|P'| < |P|$), then $\text{Osc}(f, P') \leq 3 \text{Osc}(f, P)$.

**Proof.** Let $\{x'_j\}$ be the points of $P'$ and $\{x_k\}$ the points of $P$. Look at one particular subinterval $[x'_{j-1}, x'_j]$ of $P'$. Since $|P'| < |P|$, it intersects at most two consecutive subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ of $P$.

**Case a.** Suppose $[x'_{j-1}, x'_j] \cap [x_{k-1}, x_k] \neq \emptyset$ and $[x'_{j-1}, x'_j] \cap [x_k, x_{k+1}] \neq \emptyset$. Since $M'_j = \max(M_k, M_{k+1})$ and $m'_j = \min(m_k, m_{k+1})$, we have $M'_j - m'_j \leq (M_k - m_k) + (M_{k+1} - m_{k+1})$.

Thus

$$(M'_j - m'_j)(x'_j - x'_{j-1}) \leq (M_k - m_k)(x'_j - x'_{j-1}) + (M_{k+1} - m_{k+1})(x'_j - x'_{j-1}).$$

**Case b.** Suppose that $[x'_{j-1}, x'_j]$ is contained in only one subinterval $[x_{k-1}, x_k]$ of $P$. Then

$$(M'_j - m'_j)(x'_j - x'_{j-1}) \leq (M_k - m_k)(x'_j - x'_{j-1}).$$

Both Case a and Case b imply that for any $j$

$$(M'_j - m'_j)(x'_j - x'_{j-1}) \leq \sum_{k: [x'_{j-1}, x'_j] \cap [x_{k-1}, x_k] \neq \emptyset} (M_k - m_k)(x'_j - x'_{j-1})$$

where there are at most two summands on the right hand side (RHS). Summing this inequality over all $j$, we obtain $\text{LHS}=\text{Osc}(f, P')$. And we obtain

$$\text{RHS} = \sum_{j} \sum_{k: [x'_{j-1}, x'_j] \cap [x_{k-1}, x_k] \neq \emptyset} (M_k - m_k)(x'_j - x'_{j-1}) = \sum_{k} (M_k - m_k) \times \sum_{j: [x'_{j-1}, x'_j] \cap [x_{k-1}, x_k] \neq \emptyset} (x'_j - x'_{j-1}).$$

Since $\sum_{j: [x'_{j-1}, x'_j] \cap [x_{k-1}, x_k] \neq \emptyset} (x'_j - x'_{j-1})$ is the sum of the lengths of those intervals of $P'$ covering $[x_{k-1}, x_k]$. Because their lengths are less than $(x_k - x_{k-1})$, we have

$$\sum_{j: [x'_{j-1}, x'_j] \cap [x_{k-1}, x_k] \neq \emptyset} (x'_j - x'_{j-1}) < 3(x_k - x_{k-1}).$$

Thus $\text{RHS} \leq \sum_k (M_k - m_k) \times 3(x_k - x_{k-1}) = 3 \text{Osc}(f, P)$. □

**Lemma 6.2.2** Let $\{a_j\}$ and $\{b_j\}$ be two bounded sequences such that $\lim_{j \to \infty}(a_j - b_j) = 0$ and $a_k \geq b_k$ for all $k$ and $\ell$. Then both $\{a_j\}$ and $\{b_j\}$ converge to the same limit.

**Proof.** We need only to show that $\{a_j\}$ converges. Given any limit point $a$ of $\{a_j\}$, there exists a subsequence $\{a'_{j}\}$ convergent to $a$. By the assumption, the subsequence $\{b'_{j}\}$ of $\{b_j\}$ also converges to $a$. Hence we know that the set of limit points of $\{a_j\}$ coincides with the set the limit points of $\{b_j\}$. In particular, we have $\limsup_{j \to \infty} a_j = \limsup_{j \to \infty} b_j$.

Since $a_k \geq b_\ell$ for all $k$ and $\ell$, we have $\liminf_{j \to \infty} a_j \geq \limsup_{j \to \infty} b_j$. By the previous two inequalities, we have $\liminf_{j \to \infty} a_j \geq \limsup_{j \to \infty} b_j = \limsup_{j \to \infty} a_j$. It implies $\limsup_{j \to \infty} a_j = \liminf_{j \to \infty} a_j$. Hence $\{a_j\}$ converges. □
**Theorem 6.2.1** Let \( f \) be a bounded function on \([a, b]\). Then the following are equivalent:

a. There is a sequence \( P_j \) of partitions such that \( \text{Osc} (f, P_j) \to 0 \).

b. \( \text{Osc} (f, P) \to 0 \) as \( |P| \to 0 \) in the sense that given any \( 1/n \) there exists \( 1/m \) such that \( \text{Osc} (f, P) \leq 1/n \) for any partition \( P \) with \( |P| \leq 1/m \).

c. \( \inf_P S^+(f, P) = \sup_P S^-(f, P) \).

d. There exist a sequence \( P_j \) of partitions and a real number \( \int_a^b f(x) \, dx \) such that \( S(f, P_j) \to \int_a^b f(x) \, dx \) as \( j \to \infty \) for every choice of Cauchy sums \( S(f, P_j) \).

e. \( S(f, P) \to \int_a^b f(x) \, dx \) as \( |P| \to 0 \) in the sense that given any \( 1/n \) there exists \( 1/m \) such that \( |S(f, P) - \int_a^b f(x) \, dx| \leq 1/n \) for any Cauchy sum \( S(f, P) \) and any partition \( P \) with \( |P| \leq 1/m \).

**Proof.**

a\(\iff\)b: b\(\iff\)a is obvious. We prove a\(\iff\)b by the previous lemma. Since once \( \text{Osc} (f, P_j) \leq 1/3n \), we have \( \text{Osc} (f, P) \leq 3 \text{Osc} (f, P_j) \leq 1/n \) if \( |P| < |P_j| \). Hence a\(\iff\)b.

c\(\iff\)a: Just as in the case of continuous functions, every upper sum \( S^+(f, P) \) is \( \geq \) every lower sum \( S^- (f, P') \). Thus \( \inf_P S^+(f, P) \geq \sup_P S^-(f, P) \). If we have the equality (i.e. we have c), there must exist sequences \( P_j \) and \( P_j' \) of partitions such that \( \lim_{j \to \infty} S^+(f, P_j) = \lim_{j \to \infty} S^-(f, P_j') \). Let \( P_j'' = P_j \cup P_j' \), which is the common refinement of \( P_j \) and \( P_j' \) and satisfies \( S^-(f, P_j'') \leq S^-(f, P_j') \leq S^+(f, P_j) \leq S^+(f, P_j'') \).

Thus we have \( \lim_{j \to \infty} S^+(f, P_j') = \lim_{j \to \infty} S^-(f, P_j'') \), which is the same thing as \( \lim_{j \to \infty} \text{Osc} (f, P_j'') = 0 \). Thus condition c implies condition a. But condition a implies \( \lim_{j \to \infty} S^+(f, P_j) = \lim_{j \to \infty} S^- (f, P_j') \) for a sequence \( P_j \) by Lemma 6.1.2. Hence

\[
\inf_P S^+(f, P) \leq \lim_{j \to \infty} S^+(f, P_j) = \lim_{j \to \infty} S^-(f, P_j') \leq \sup_P S^-(f, P).
\]

So \( \inf_P S^+(f, P) = \sup_P S^-(f, P) \).

d\(\iff\)a: Since the oscillation of \( f \) on \( P \) is the spread of the values of the Cauchy sums \( S(f, P) \), the convergence of \( S(f, P_j) \) implies \( \text{Osc} (f, P_j) \to 0 \). So condition d implies condition a.

e\(\iff\)d is obvious.

c\(\iff\)e: Let \( \int_a^b f(x) \, dx = \inf_P S^+(f, P) = \sup_P S^-(f, P) \). We need only to show that \( S^-(f, P) \to \int_a^b f(x) \, dx \) and \( S^+(f, P) \to \int_a^b f(x) \, dx \) as \( |P| \) tends to zero. We only prove the former (by contradiction argument). The proof of the latter is similar. The negation of the former is: there exists a sequence \( P_j \) of partitions and positive integer \( n_0 \) such that \( |P_j| \to 0 \) and for all \( j \)

\[
| \sup_P S^-(f, P) - S^-(f, P_j) | > 1/n_0 \Rightarrow \sup_P S^-(f, P) > S^-(f, P_j) + 1/n_0.
\]

Since c\(\iff\)a\(\iff\)b and \( |P_j| \to 0 \), we have \( \text{Osc} (f, P_j) \to 0 \) by b, so by Lemma 6.1.2 we obtain \( \lim_{j \to \infty} S^+(f, P_j) = \lim_{j \to \infty} S^- (f, P_j) = 0 \). As \( j \to \infty \) the previous inequality gives

\[
\sup_P S^-(f, P) \geq \lim_{j \to \infty} S^-(f, P_j) + 1/n_0 = \lim_{j \to \infty} S^+(f, P_j) + 1/n_0 \geq \inf_P S^+(f, P) + 1/n_0,
\]

a contradiction. \( \square \)
In the class of 04/06 (Wed) I stated a lemma on oscillation of functions without detailed proof. Here I give its proof.

**Lemma (Oscillations of functions)** Let \( f \) and \( g \) be two bounded functions on \([a, b]\), and \( P \) be a partition of \([a, b]\). Then

a. \( \text{Osc} \left( f + g, P \right) \leq \text{Osc} \left( f, P \right) + \text{Osc} \left( g, P \right) \).

b. \( \text{Osc} \left( cf, P \right) = |c| \text{Osc} \left( f, P \right) \), where \( c \) is a constant.

c. \( \text{Osc} \left( |f|, P \right) \leq \text{Osc} \left( f, P \right) \).

d. \( \text{Osc} \left( \max(f, g), P \right) \leq \text{Osc} \left( f, P \right) + \text{Osc} \left( g, P \right) \). \( \text{Osc} \left( \min(f, g), P \right) \leq \text{Osc} \left( f, P \right) + \text{Osc} \left( g, P \right) \).

e. \( \text{Osc} \left( f \cdot g, P \right) \leq M \left( \text{Osc} \left( f, P \right) + \text{Osc} \left( g, P \right) \right) \), where \( M \) is the sup of \(|f| + |g|\) on \([a, b]\).

**Proof.** Let \( M_k(f), m_k(f) \) be the sup and inf of \( f \) on the subinterval \([x_{k-1}, x_k]\), resp. Recall \( \text{Osc} \left( f, P \right) = \sum_{k=1}^{n} \left( M_k(f) - m_k(f) \right) (x_k - x_{k-1}) \). We only need to prove the corresponding inequalities for the quantity \( M_k(f) - m_k(f) \). Let \( x \) and \( y \) be two points in the interval.

a. Then by the triangle inequality

\[
| (f + g)(x) - (f + g)(y) | \leq |f(x) - f(y)| + |g(x) - g(y)| \leq (M_k(f) - m_k(f)) + (M_k(g) - m_k(g)),
\]

so \( M_k(f + g) - m_k(f + g) \leq (M_k(f) - m_k(f)) + (M_k(g) - m_k(g)) \).

b follows from \( M_k(cf) - m_k(cf) = |c| (M_k(f) - m_k(f)) \).

c. By the triangle inequality,

\[
||f(x)| - |f(y)||| \leq |f(x) - f(y)|, \quad \text{so} \quad M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f).
\]

d. Recall \( \max(f, g) = (f + g + |f - g|)/2 \). By a-c, we have

\[
\text{Osc} \left( \max(f, g), P \right) = \text{Osc} \left( \left( f + g + |f - g| \right)/2, P \right) \leq \frac{1}{2} \left( \text{Osc} \left( f + g, P \right) + \text{Osc} \left( |f - g|, P \right) \right) \leq \frac{1}{2} \left( \left( \text{Osc} \left( f, P \right) + \text{Osc} \left( g, P \right) \right) \right) \leq \text{Osc} \left( f, P \right) + \text{Osc} \left( g, P \right).
\]

e is included in the homework.

**Theorem 6.2.2** Let \( f \) and \( g \) be Riemann integrable on \([a, b]\). Then the following hold:

a. \( |f| \) is Riemann integrable and \( \int_a^b |f(x)| \, dx = \int_a^b |f(x)| \, dx \).

b. \( \max(f, g) \) and \( \min(f, g) \) is Riemann integrable.

c. \( f \cdot g \) is Riemann integrable. (HW)
Proof. a. The Riemann integrability of $|f|$ follows from statement c of the previous lemma and the Osc $(f, P_j) \to 0$ criterion (Condition a of Theorem 6.2.1). Since $-|f(x)| \leq f(x) \leq |f(x)|$, we have

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx$$

by the order preservation of the integral, so we obtain $\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx$.

b. It follows from Statement d of the previous lemma and the above criterion. □

Homework (04/04-04/06)
§ 6.1.5: 2-4, 9, 10, 13; § 6.2.4: 2-4, 9, 11; reading § 6.3 and § 7.1-2.