Solutions for Some Exercises of Chapter 3

Exercises 3.1.3 (page 84)

2. The answer of the first question is NO. A counterexample is that the sequence 0, 1, 0, 1, ⋮ can be decomposed into the sum of the monotone increasing sequence 2, 3, 4, 5, ⋮ and the monotone decreasing sequence −2, −2, −4, −4, ⋮.

If a monotone increasing sequence \( \{y_n\} \) and a monotone decreasing sequence \( \{z_n\} \) are bounded, then they converge and their sum \( \{y_n + z_n\} \) also converges.

5. Suppose both \( \limsup \{x_n\} \) and \( \limsup \{y_n\} \) are finite. We show the inequality

\[
\limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}.
\]

We only need to show that if \( \{x'_n + y'_n\} \) is a subsequence of \( \{x_n + y_n\} \) and converging to \( x \), then \( x \) is \( \leq \limsup\{x_n\} + \limsup\{y_n\} \). Since the sequence is bounded from above, we may assume \( x \) is a real number without loss of generality. Since

\[
x'_{n+1} + y'_{n+1} \leq \sup_{k>n} x_k + \sup_{k>n} y_k,
\]

we have \( x \leq \limsup\{x_n\} + \limsup\{y_n\} \) since limits preserve non-strict inequalities.

7. The statement of this question is not rigorous. "limit-points" here should be replaced by "finite limit-points" because a sequence with any integer as its limit-point must be unbounded both from above and from below and then have \( \pm\infty \) as its limit-points. For any integer \( k \), the sequence \( \{k + 1/n\} \) converges to \( k \).

**STEP 1**

By diagonalizing the rectangular array

\[
\begin{array}{cccccc}
0 + 1 & 0 + 1/2 & 0 + 1/3 & \cdots \\
1 + 1 & 1 + 1/2 & 1 + 1/3 & \cdots \\
2 + 1 & 2 + 1/2 & 2 + 1/3 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

by Exercise 3.1.3:11, we obtain a sequence \( x_1, x_2, \cdots \) which has limit-points exactly as \( +\infty \) and non-negative integers. Indeed, any subsequence \( x'_1, x'_2, \cdots \) has the form \( x'_j = a_j + 1/b_j \), where \( a_j \) is a non-negative integer, \( b_j \) is a positive integer, and either \( \{a_j\} \) or \( \{b_j\} \) has the limit \( +\infty \). Hence the subsequence has a limit of a non-negative integer or \( +\infty \).

**STEP 2**

Similarly, we can also construct a sequence \( y_1, y_2, \cdots \) which has limit-points exactly as \( -\infty, -1, -2, \cdots \). Then the shuffled sequence \( x_1, y_1, x_2, y_2, \cdots \) has limit-points exactly as \( \pm\infty \) and integers.

11. Since any subsequence of any row or column of the array is also a subsequence of the sequence, any limit-point of any row or column of the array is a limit-point of the sequence. However, we cannot get all limit-points of the sequence this way. A counterexample is the array \( (a_{mn}) = (n - m) \) (Why).
Exercises 3.2.3 (page 98)

6. Let $A$ be an infinite set. We construct a countable and dense subset $B$ of $A$ as follows: for any rational number $r$ and for any positive integer $n$, we choose a number $x_n$ in $A$ for $B$ such that $|x_n - r| < 1/n$ if there are any. Then $B$ is a countable set which is dense in $A$ (Why). You may have found that the construction of the dense subset $B$ is similar to that of a countable subcover in the proof of Theorem 3.3.2 (page 104 of the textbook).

8. The statement in this question is not rigorous because $\pm \infty$ may be limit-points of a sequence, but any closed set is just a subset of real numbers and cannot contain either of $\pm \infty$. The rigorous form should be

The set of finite limit-points of a sequence of real numbers is a closed set.

**Proof.** Let $A$ be the set of finite limit-points of a sequence $x_1, x_2, \cdots$ and $x$ be a limit-point of $A$ (recall the definition of limit-points of a set of real numbers, $x$ must be a finite real number). Then there exists a sequence $y_1, y_2, \cdots$ of distinct points in $A$ such that $|y_n - x| < 1/n$ for all $n \in \mathbb{N}$. Since $y_n$ is a limit-point of the sequence, there are an infinite number of terms $x_j$ satisfying $|y_n - x_j| < 1/n$. By the triangle inequality, these infinite terms $x_j$ satisfy $|x_j - x| < 2/n$ (the factor 2 here is not essential). Therefore, $x$ is also a limit-point of the sequence.

**Remark:** The statement above immediately gives a negative answer to Exercise 3.1.3:9.

**Remark:** The set of limit-points of a set of real numbers is a closed set.

12a. Recall the Cantor set $C$ consists of all numbers in $[0, 1]$ which can be expressed with just 0’s and 2’s by the base-3 notation. We first prove that $C$ is a perfect set. Since we have known that it is closed, we only need to show that every point of $C$ is its limit-point. For a point $x$ in $C$, there are only two possibilities as follows:

Case a. $x$ can be expressed by the base-3 notation as $x = 0.a_1a_2\cdots a_k$ with $a_k = 2$. Then the sequence $0.a_1a_2\cdots a_k2, 0.a_1a_2\cdots a_k02, 0.a_1a_2\cdots a_k002, \cdots$ of distinct points in $C$ converge to $x$. Hence $x$ is a limit-point of $C$.

Case b. $x$ has no finite expansion by the base-3 notation, that is, $x = 0.b_1b_2b_3\cdots$, where there are an infinite number of terms $b_j$ equal to 2. Then the sequence $0.b_1, 0.b_1b_2, 0.b_1b_2b_3, \cdots$ containing an infinite number of points of $C$ converge to $x$. Hence $x$ is also a limit-point of $C$.

Then we prove that $C$ contains no interval. If not, $C$ must contain a closed interval $[k/3^n, (k+1)/3^n]$ for some integer $k$ when $n$ is large enough. However, at least the middle-third of the interval has been removed by stage $(n+1)$ of the construction process of $C$.

14. The answer is that only the empty set and the whole real line can be both open and closed.

**Proof.** Let $A$ be a non-empty set which is both open and closed. Then its complement $A'$ is also both open and closed. We prove $A'$ is the empty set. Suppose not. Then there exists a point $x \in A$ and another point $y \in A'$. Without loss of generality we may assume $x < y$. Since $A$ is open, $(x-1/n, x+1/n)$ is contained in $A$ when $n$ is large. Then the set of numbers in $A$ larger than $x$ and smaller than $y$, which we call $A_x$, is nonempty. Since $A$ is also closed, we have $\sup A_x \in A$ (Why). So $\sup A_x < y$. Since $A$ is open, $(\sup A_x - 1/m, \sup A_x + 1/m)$ is contained in $A$ when $m$ is large. Hence there exists a number in $A$ larger than $\sup A_x$. Contradict the definition of $\sup A_x$.
Let $O$ be a non-empty open set. Then $O$ can be expressed by the union of "maximal open intervals" containing points of $O$, where we define the maximal open interval containing a point $x$ in $O$ to be $(a_x, b_x)$ with

$$a_x = \inf\{a : (a, x] \subseteq O\} \quad (\text{may be } -\infty), \quad b_x = \sup\{b : [x, b) \subseteq O\} \quad (\text{may be } +\infty).$$

That is, $O = \bigcup_{x \in O} (a_x, b_x)$. Then any two maximal intervals coincide if they intersect. (Why). We have proved in the class that any set of disjoint open intervals is at most countable. Hence we can express $O$ in a unique way as $O = \bigcup (a_n, b_n)$, where $n$ varies in a subset of $\mathbb{Z}$ and $a_n < b_n < a_{n+1} < b_{n+1}$ for all $n$.

**Exercises 3.3.1 (page 106)**

3. HINT: What sets are both open and closed?

6. Let $A$ and $B$ be non-empty sets. If $A$ is open, then $A + B$ is the union of open sets $A + \{x\}$ where $x$ varies in $B$. Hence $A + B$ is open.

Let $A$ and $B$ be nonempty compact sets. To prove $A + B$ is compact, we only need to show that $A + B$ is bounded and closed. That $A + B$ is bounded is obvious. And that $A + B$ is closed follows from the following fact:

*Let $C$ be a compact set and $D$ be a closed set. Then $C + D$ is a closed set*.†

**Proof.** Let $x$ be a limit-point of $C + D$. Then there exists a sequence $x_1, x_2, \cdots$ of points in $C + D$ convergent to $x$. We can write $x_n = c_n + d_n$, where $c_n \in C$ and $d_n \in D$. Since $\{c_n\}$ is a sequence in the compact set $C$, there is a subsequence $\{c'_n\}$ convergent to a point $c$ in $C$. Then the corresponding subsequence $\{d'_n\}$ of $\{d_n\}$ also converges to the point $x - c$.

Case 1. Suppose that the sequence $\{d'_n\}$ of points in $D$ only contains finite number of points. Then except finite terms all the $d'_n$’s are constant $x - c$ in $D$. Hence $x = (x - c) + c$ is in $C + D$.

Case 2. Suppose that $\{d'_n\}$ contains an infinite number of points in $D$. Then $x - c$ is a limit-point of $D$ and then is in $D$ since $D$ is closed. We also have $x \in C + D$. \qed

An example where $A$ and $B$ are closed but $A + B$ is not is as follows: $A = \mathbb{N}$, $B = \{-n + 1/n : n \in \mathbb{N}\}$ (Why).

†There is an alternative proof that $A + B$ is compact by showing every open cover of $A + B$ has a finite subcover. It is more complicated but seems more interesting.